A Farey Tour

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Images: Sage Mathematics Software, Ipe, xypic

The Farey subdivision

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

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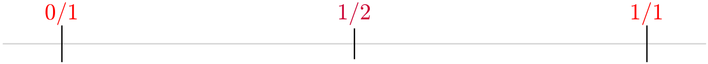

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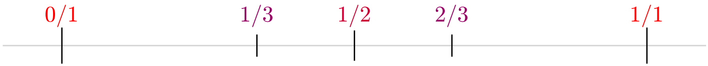

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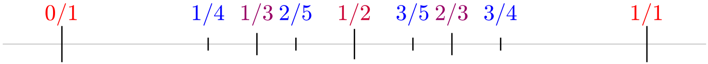

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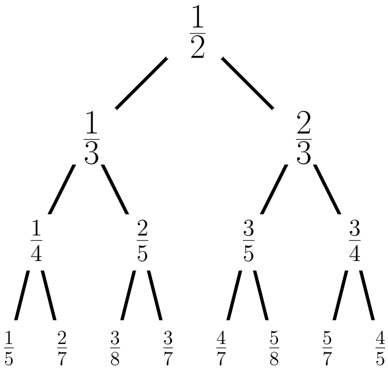
An arborist's view of $\mathbb{P}^1(\mathbb{Z})$



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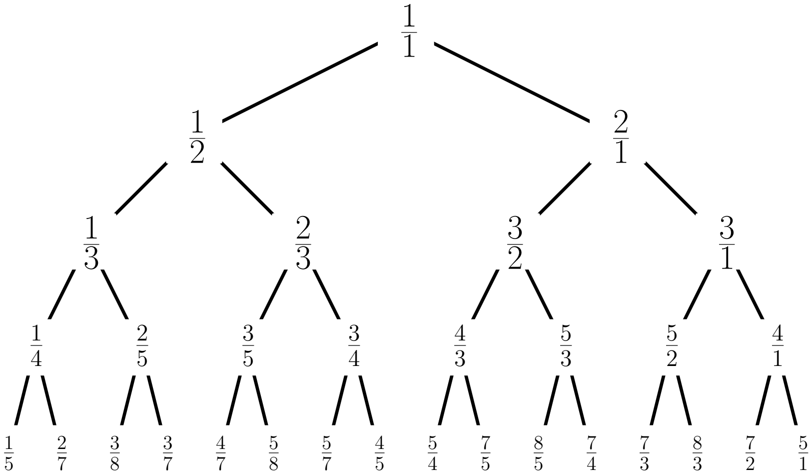
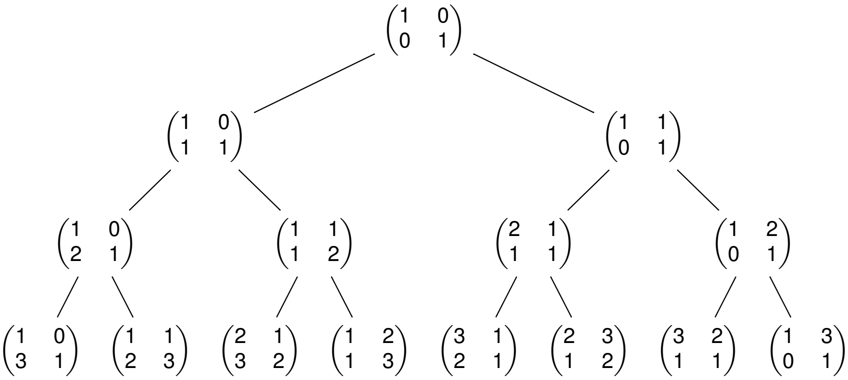
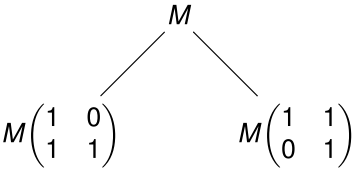


Money may not, but matrices do: $\mathrm{SL}_2^+(\mathbb{Z})$





Money may not, but matrices do: $SL_2^+(\mathbb{Z})$

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\color{red}{\begin{matrix}1\\1\end{matrix}}$$

$$\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\color{red}{\begin{matrix}1\\2\end{matrix}} \qquad \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\color{red}{\begin{matrix}2\\1\end{matrix}}$$

$$\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\color{red}{\begin{matrix}1\\3\end{matrix}} \quad \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\color{red}{\begin{matrix}2\\3\end{matrix}} \quad \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\color{red}{\begin{matrix}3\\2\end{matrix}} \quad \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\color{red}{\begin{matrix}3\\1\end{matrix}}$$

$$\begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}\color{red}{\begin{matrix}1\\4\end{matrix}} \begin{pmatrix}1 & 1\\ 2 & 3\end{pmatrix}\color{red}{\begin{matrix}2\\5\end{matrix}} \begin{pmatrix}2 & 1\\ 3 & 2\end{pmatrix}\color{red}{\begin{matrix}3\\5\end{matrix}} \begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}\color{red}{\begin{matrix}3\\4\end{matrix}} \begin{pmatrix}3 & 1\\ 2 & 1\end{pmatrix}\color{red}{\begin{matrix}4\\3\end{matrix}} \begin{pmatrix}2 & 3\\ 1 & 2\end{pmatrix}\color{red}{\begin{matrix}5\\3\end{matrix}} \begin{pmatrix}3 & 2\\ 1 & 1\end{pmatrix}\color{red}{\begin{matrix}5\\2\end{matrix}} \begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}\color{red}{\begin{matrix}4\\1\end{matrix}}$$

$$M \mapsto \color{red}{M\begin{pmatrix}1\\1\end{pmatrix}}$$

The address of $\alpha \in \mathbb{R}$

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

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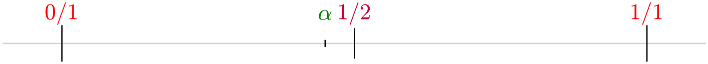

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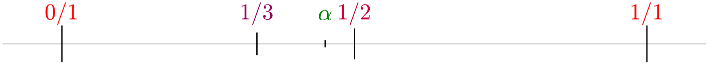

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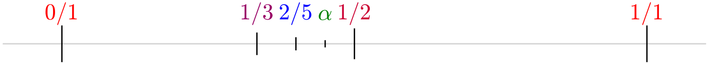
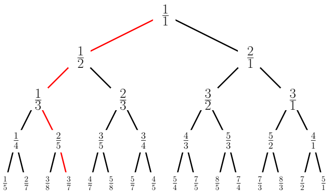


$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

The address of α

Real number α



Infinite path through tree:

$$L^{a_0} R^{a_1} L^{a_2} R^{a_3} \ldots$$

Matrix product:

$$\begin{pmatrix} 1 & 0 \\ a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_3 \\ 0 & 1 \end{pmatrix} \ldots$$

The Farey subdivision: Continued fractions / Euclidean algorithm

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}}$$

$$\begin{aligned} x_{n+2} &= -x_{n+1} a_n + x_n \\ &\vdots \\ x_4 &= -x_3 a_2 + x_2 \\ x_3 &= -x_2 a_1 + x_1 \\ x_2 &= -x_1 a_0 + x_0 \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} p_n \\ q_n \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & a_0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_3 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_3 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

Farey subdivision: frothy version

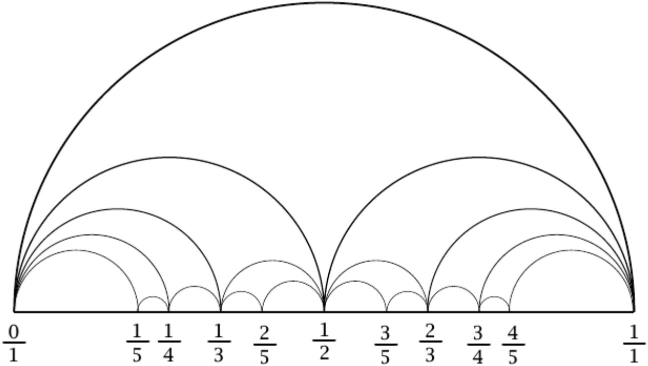


image from Allen Hatcher's *Topology of Numbers*

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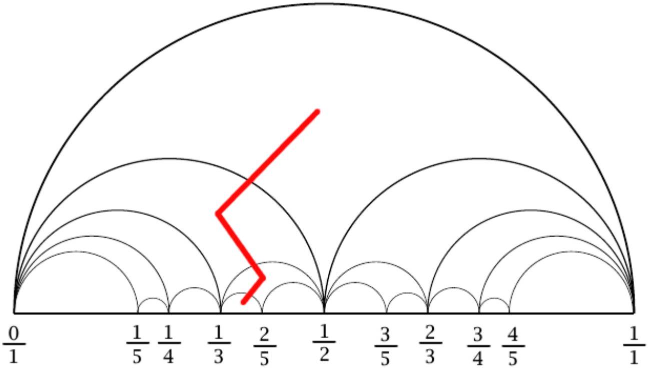

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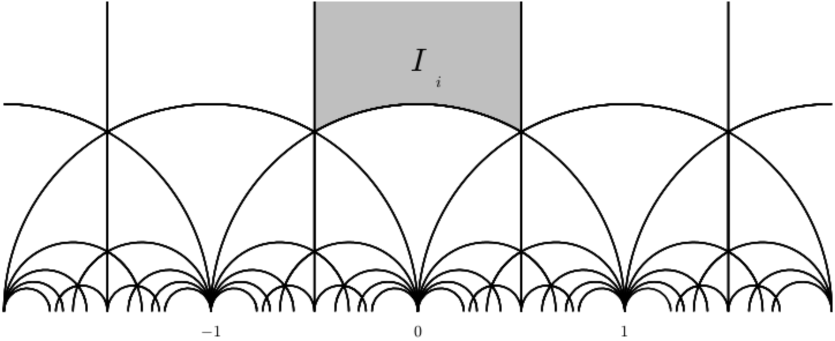
$\mathrm{PSL}_2(\mathbb{Z})$

$$\left[z \mapsto \frac{az+b}{cz+d}\right] \quad \leftrightsquigarrow \quad \left[\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right]$$



$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

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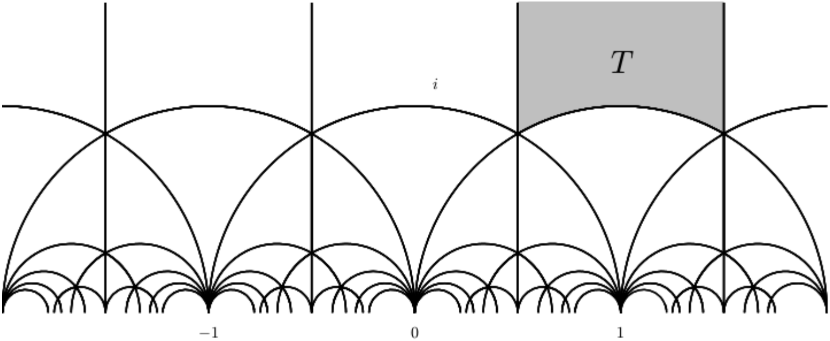

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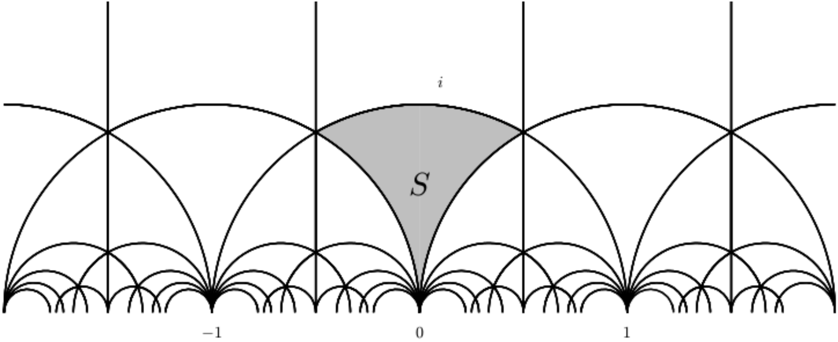

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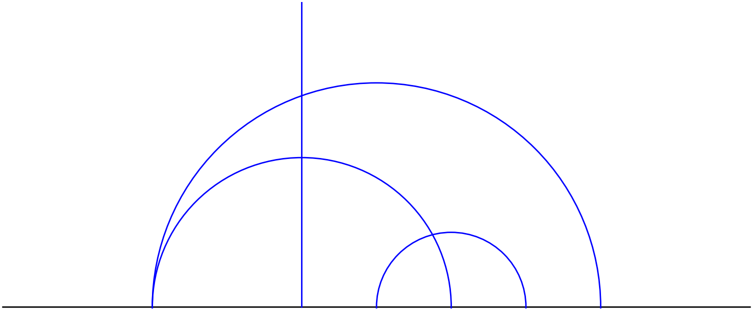
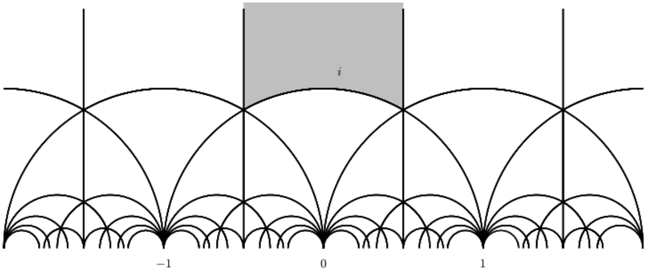
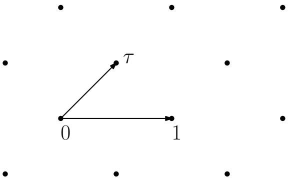
Hyperbolic 2-space



Moduli space of elliptic curves

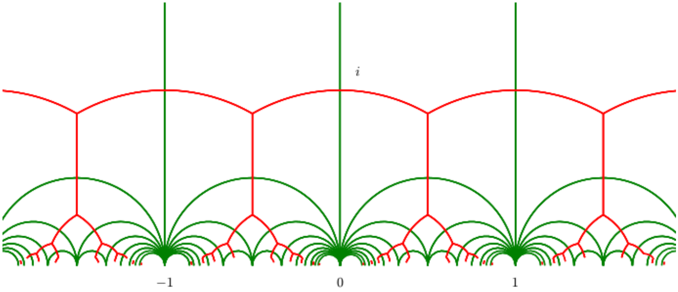




$\{\text{elliptic curves over } \mathbb{C}\} \longleftrightarrow \{\mathbb{C}/\Lambda\}$

$SL_2(\mathbb{Z}) = \text{change of basis}$

Tree in Bubbles



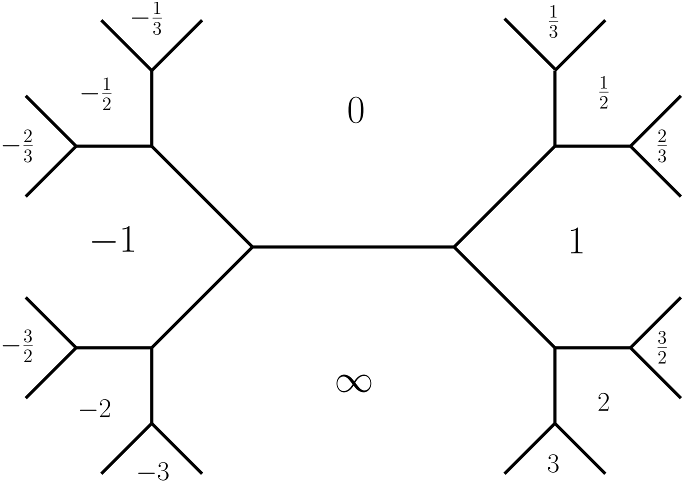
Tesselation group:

$$\Pi(2) = \{M \in \mathrm{PGL}_2(\mathbb{Z}) : M \cong I \pmod 2\}$$
$$= \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle .$$

Cayley graph / Stern-Brocot tree (red):

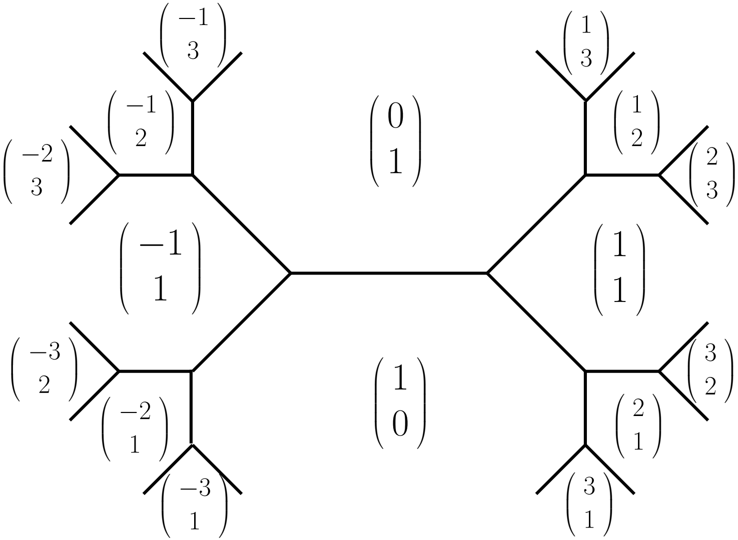
- Vertices: triangles / elements of $\Pi(2)$
- edges: walls / pairs (g, gs) where s is a generator

Topograph

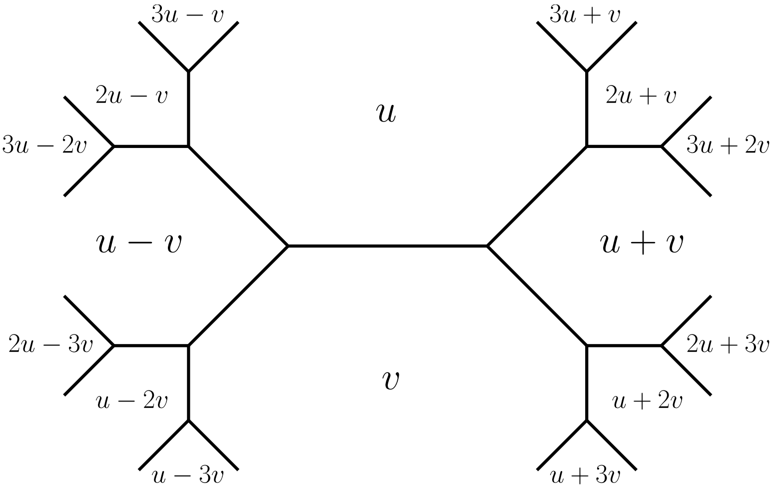


$-\frac{1}{3}$
$\frac{1}{3}$
$-\frac{1}{2}$
0
$\frac{1}{2}$
$-\frac{2}{3}$
$\frac{2}{3}$
-1
1
$-\frac{3}{2}$
$\frac{3}{2}$
∞
-2
2
-3
3

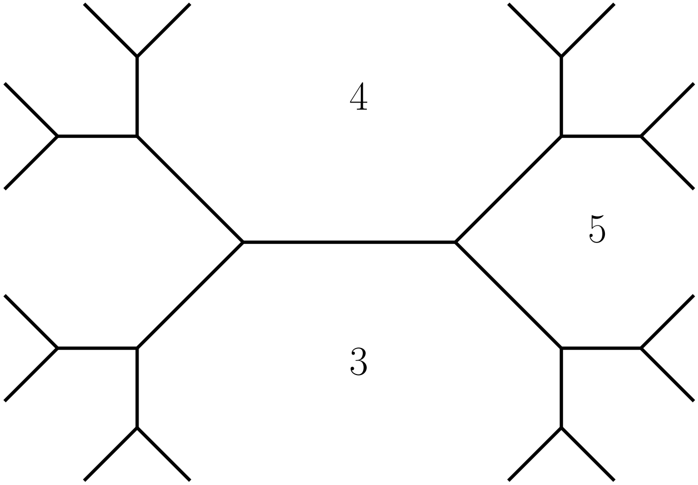
vertex = superbasis (3 vectors, each pair a basis)



$f(u+v) + f(u-v) = 2(f(u) + f(v))$



Conway's sensual quadratic form



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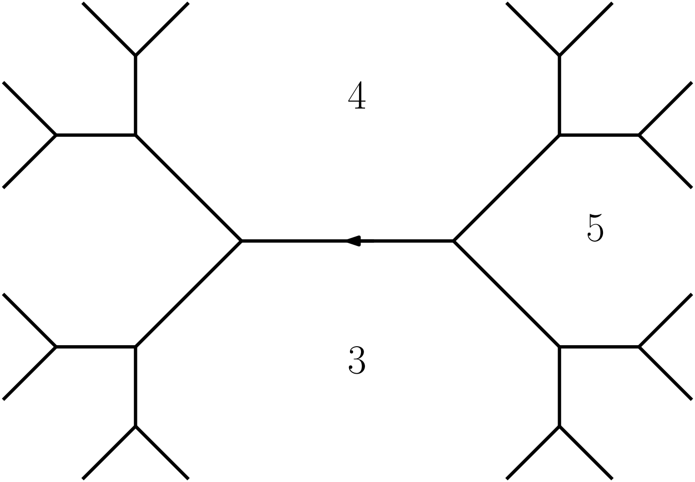

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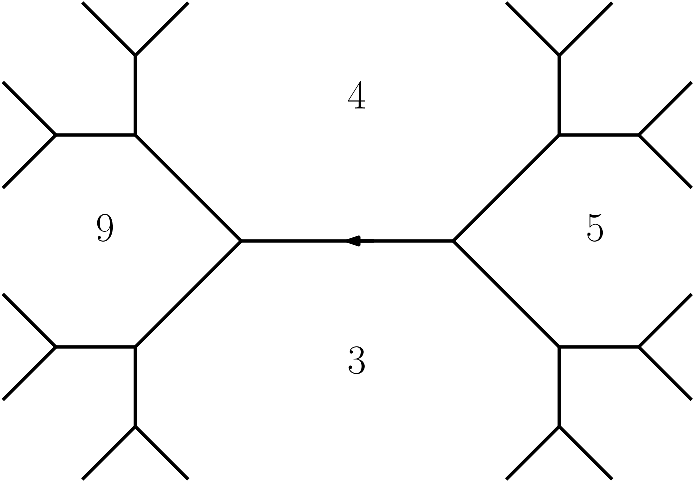

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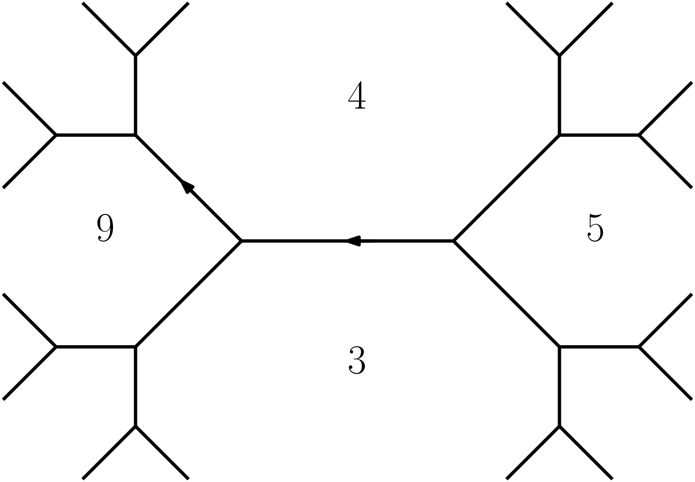

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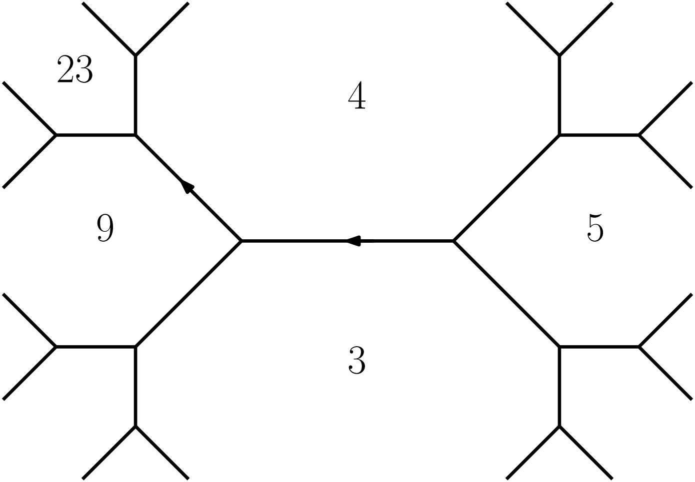

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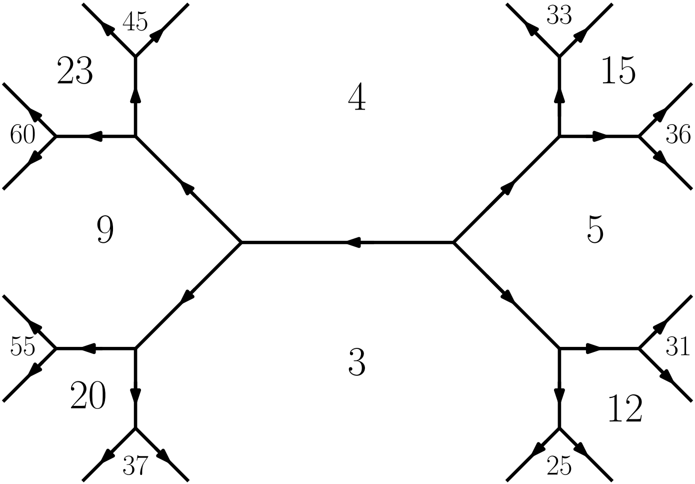

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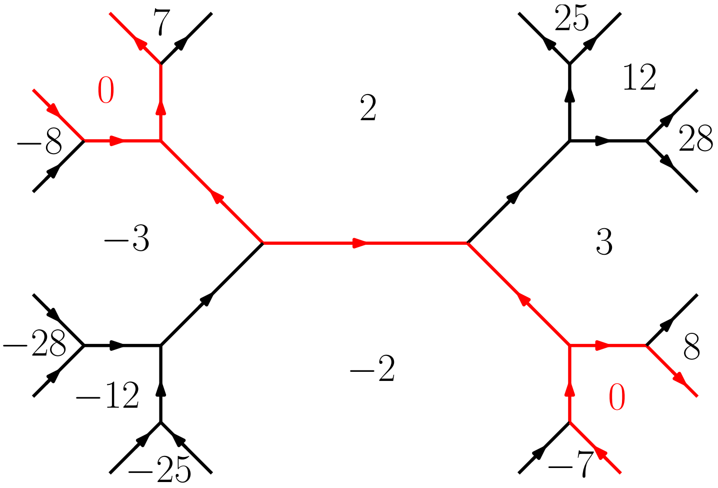
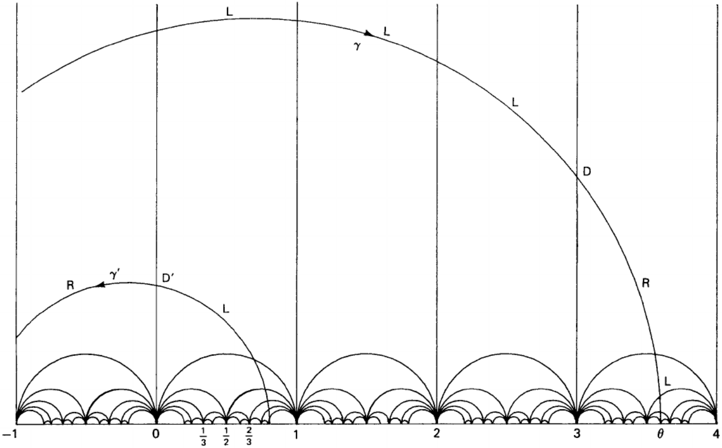


Continued fractions as geodesics



address of α = $L^{a_0} R^{a_1} L^{a_2} R^{a_3} \cdots$ = $[a_0, a_1, a_2, \ldots]$

Image from Caroline Series' *The Geometry of Markoff Numbers*

Gauss map

Question

For $\alpha \in \mathbb{R}$, what is frequency of appearance of a finite string $n_0, n_1, n_2, \ldots, n_k$ in the continued fraction expansion?

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Answer

$$\frac{1}{\ln 2} \int_{(a,b)} \frac{dx}{1+x} = \log_2 \left(\frac{1+b}{1+a} \right)$$

where (a, b) is the interval of real numbers with continued fraction expansion beginning $n_0, \ldots, n_k$.

E.g. with probability one, α has continued fraction expansion with frequency $0.415\ldots$ of the digit 1.

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Reason

$$T : x \mapsto \{1/x\} \quad \textit{(fractional part)}$$

T has invariant measure $\frac{1}{\ln 2} \frac{dx}{1+x}$ w.r.t. which it is ergodic

Farey Tesselation

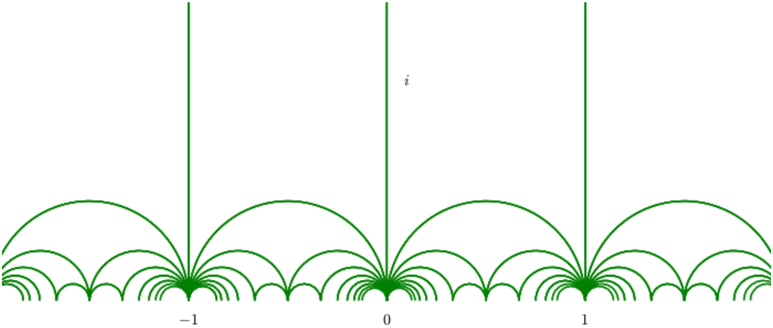


Image of $\{0, \infty\}$ (and its geodesic) under $\mathrm{PSL}_2(\mathbb{Z})$.

Schmidt Arrangements

The *Schmidt arrangement* of a imaginary quadratic field K is the orbit of $\widehat{\mathbb{R}}$ under the Möbius transformations given by the *Bianchi group*

$$\mathrm{PSL}_2(\mathcal{O}_K) = \left\{ \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} : \alpha, \beta, \delta, \gamma \in \mathcal{O}_K,\ \alpha\delta - \beta\gamma = 1 \right\} / \pm I$$

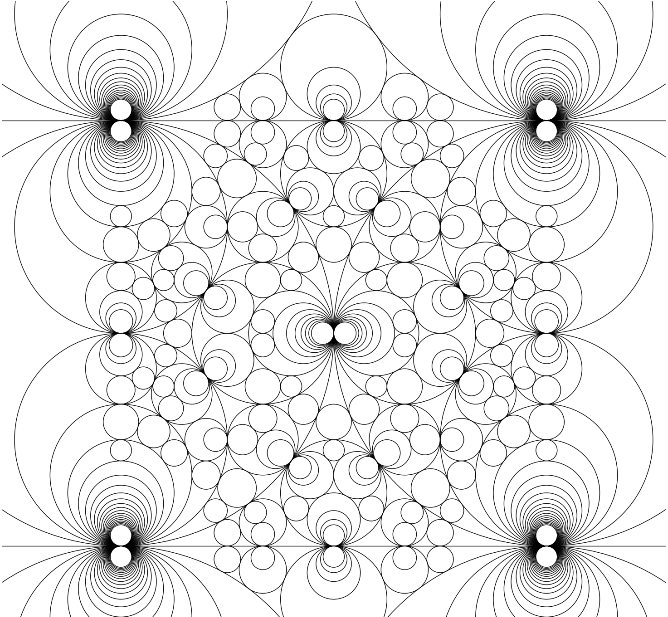
That is,

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \leftrightarrow \left(z \mapsto \frac{\alpha z + \gamma}{\beta z + \delta} \right).$$

Each individual image $M(\widehat{\mathbb{R}})$ is called a *K-Bianchi circle*.

$$\mathcal{S}_K = \{K\text{-Bianchi circles}\}$$

Schmidt Arrangement of $\mathbb{Q}(i)$



History: 1975

DIOPHANTINE APPROXIMATION OF COMPLEX NUMBERS 5

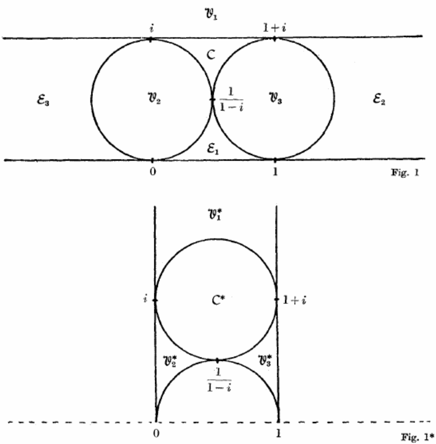


Fig. 1

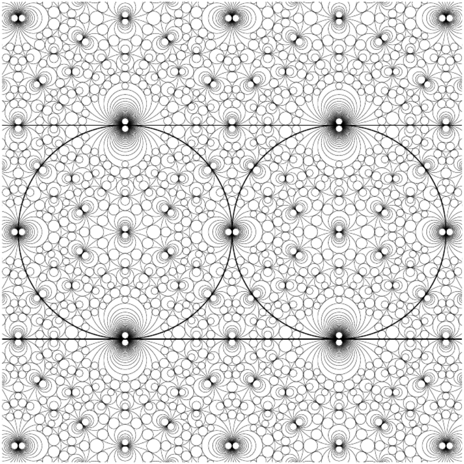
Fig. 1*

- Asmus Schmidt, *Diophantine Approximation of Complex Numbers*, Acta Arithmetica, 1975.
- Continued fractions for $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-2}]$ etc. made use of $\mathcal{S}_K$ (defined essentially this way).

Comparison

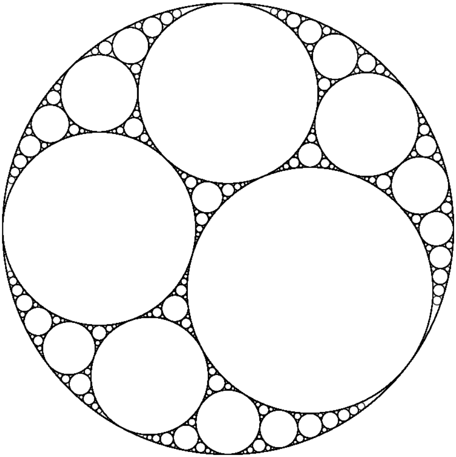
$\mathbb{Z}$	$\mathbb{Z}[i]$
$\mathrm{PSL}_2(\mathbb{Z})$	$\mathrm{PSL}_2(\mathbb{Z}[i])$
$\{0, \infty\}$	$\widehat{\mathbb{R}}$
$\Pi(2)$	?
superbasis (3 pts)	?
swap out one	?
Stern-Brocot tree	?
Gauss map	?

History: 2006



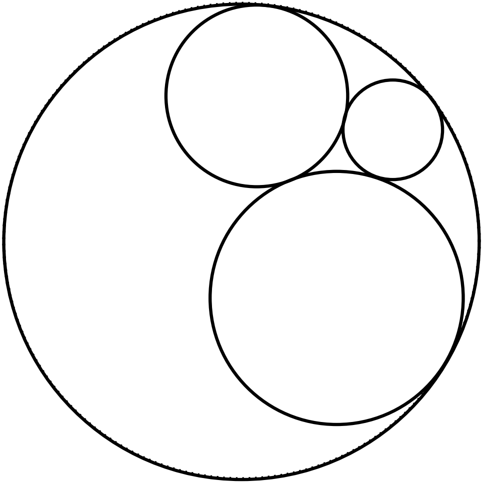
- Graham, Lagarias, Mallows, Wilks, Yan, *Apollonian Circle Packings: Geometry and Group Theory II. Super-Apollonian Group and Integral Packings*, Discrete and Computational Geometry, 2006.
- Superpacking defined differently but equivalently.

Apollonian Circle Packings



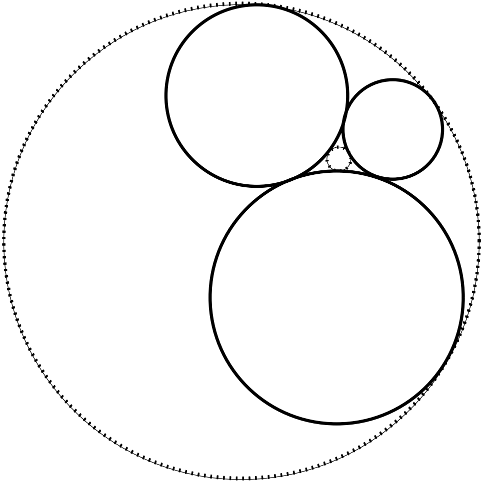
Apollonian Circle Packings

A Descartes quadruple is any collection of four circles which are pairwise mutually tangent, with disjoint interiors.



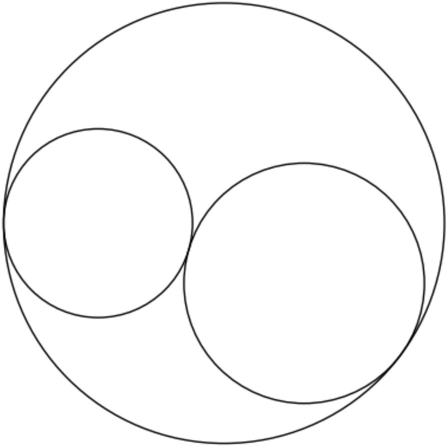
Apollonian Circle Packings

Given any three mutually tangent circles, there are exactly two ways to complete the triple to a Descartes quadruple.



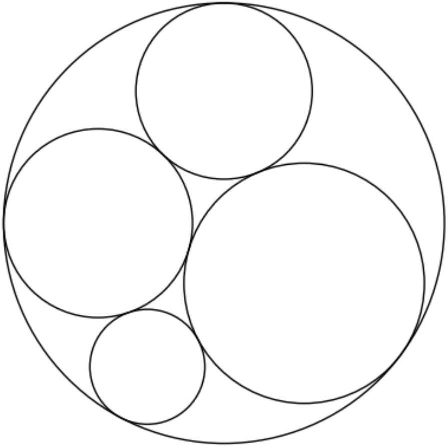
Apollonian Circle Packings

Beginning with any three mutually tangent circles...



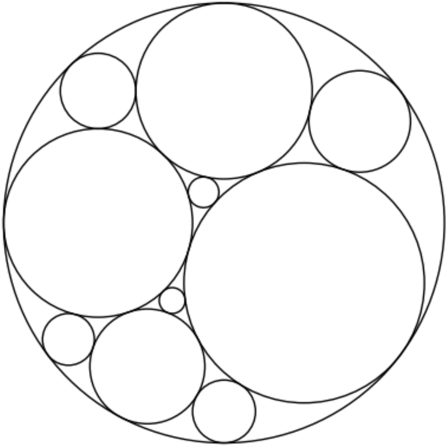
Apollonian Circle Packings

Beginning with any three mutually tangent circles, add in both new circles which would complete the triple to a Descartes quadruple.



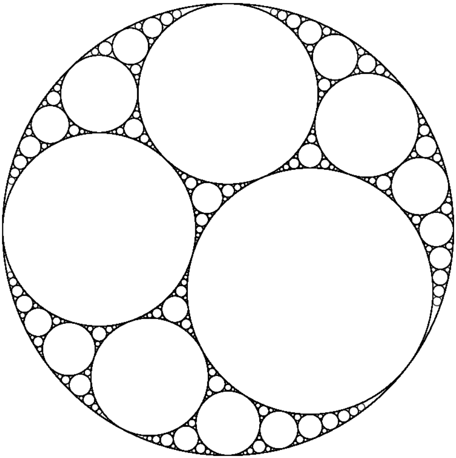
Apollonian Circle Packings

Repeat: for every triple of mutually tangent circles in the collection, add the two ‘completions.’



Apollonian Circle Packings

Repeating ad infinitum, we obtain an Apollonian circle packing.



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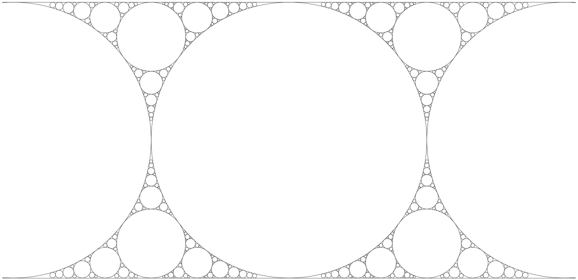
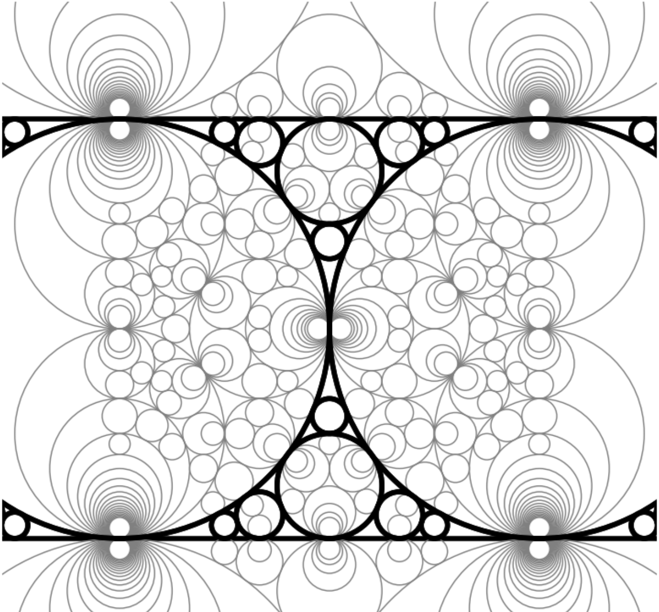
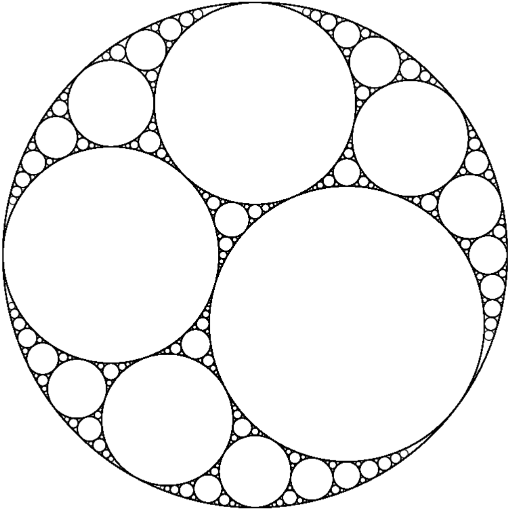
Repeating ad infinitum, we obtain an Apollonian circle packing.



ACP in Schmidt Arrangement of $\mathbb{Q}(i)$



Apollonian Circle Packings



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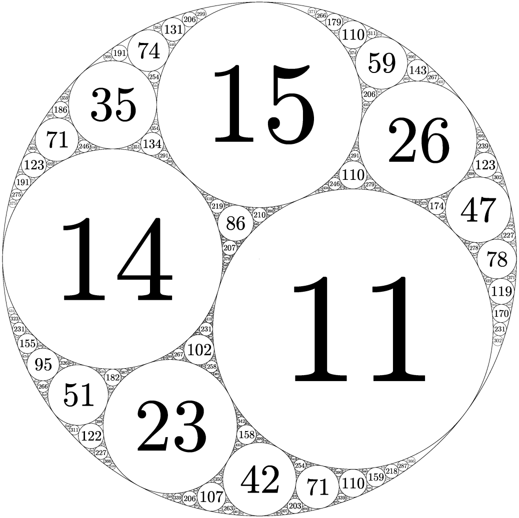
Apollonian Circle Packings



The outer circle has curvature -6 (its interior is outside).

The Descartes Rule

The curvatures (inverse radii) in a Descartes configuration satisfy

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

If a, b, c are fixed, there are two solutions d, d', where

$$d + d' = 2(a + b + c).$$

Hence an **integer** Descartes quadruple generates an Apollonian packing of **integer curvatures.**

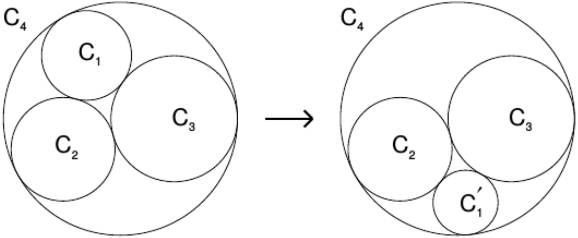
Local-Global Conjecture

Conjecture (Graham–Lagarias–Mallows–Wilks–Yan, Fuchs–Sanden)

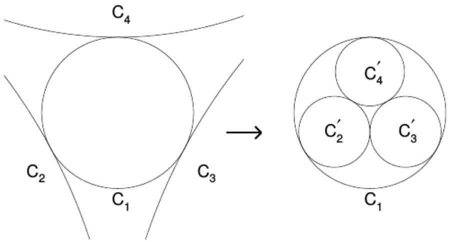
$\mathcal{P}$ a primitive, integral ACP. Let S be the set of residues of curvatures modulo 24*. Then any sufficiently large integer with a residue in S occurs as a curvature.*

- Bourgain, Fuchs: Curvatures have positive density in $\mathbb{Z}$.
- Bourgain, Kontorovich: Density one occur.

Super-Apollonian group



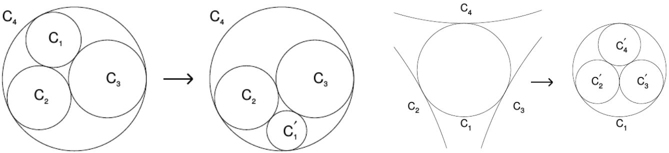
(4 swaps)



(4 inversions)

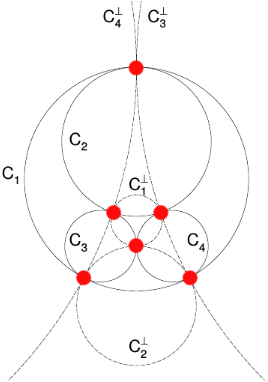
Images from Graham, Lagarias, Mallows, Wilks, Yan

Super-Apollonian group



$$\left\langle S_1, S_2, S_3, S_4, S_1^{\perp}, S_2^{\perp}, S_3^{\perp}, S_4^{\perp} : \right.$$
$$\left. S_i^2 = (S_i^{\perp})^2 = (S_i S_j^{\perp})^2 = (S_j^{\perp} S_i)^2 = 1 \right\rangle$$

Comparison



$\mathbb{Z}$	$\mathbb{Z}[i]$
$PSL_2(\mathbb{Z})$	$PSL_2(\mathbb{Z}[i])$
$\{0, \infty\}$	$\widehat{\mathbb{R}}$
$\Pi(2)$	super-Apollonian grp
superbasis (3 pts)	Descartes quad'ple with dual (6 pts)
swap out one	swap out three
Stern-Brocot tree	?
Gauss map	?

Image modified from Graham, Lagarias, Mallows, Wilks, Yan

Lorentz form

Denote a Descartes quadruple by its curvatures (A, B, C, D). The Descartes form

$$2(A^2 + B^2 + C^2 + D^2) = (A + B + C + D)^2$$

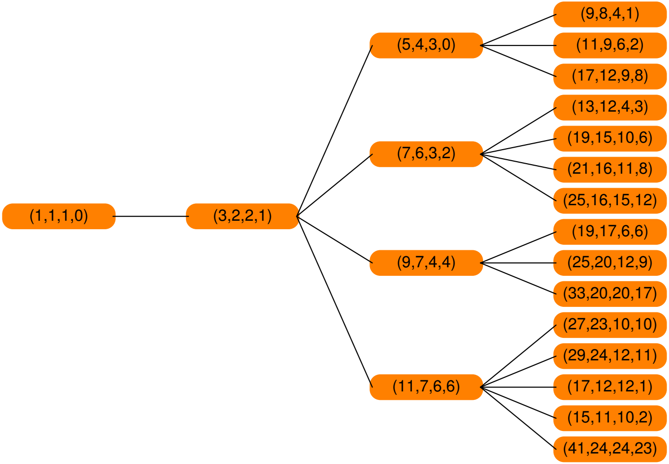
is conjugate to the Lorentz form

$$a^2 = b^2 + c^2 + d^2.$$

So we may also use a Lorentz quadruple (a, b, c, d).

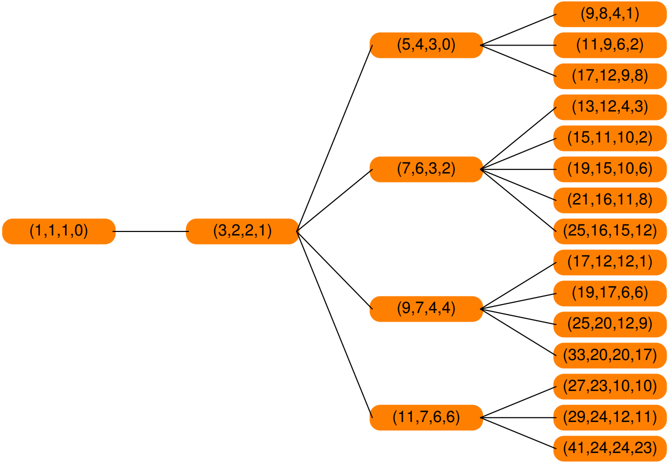
Swap down tree

Descend the Cayley graph using *normal form* (Graham et al.): No S_i^2, $(S_i^\perp)^2$ or $S_i^\perp S_j$. In other words, descend by a swap instead of an inversion if possible.



Drop down tree

Descend to origin, by repeatedly moving to the quadruple whose Lorentz form (a, b, c, d) has smallest a.



Swap down dynamical system

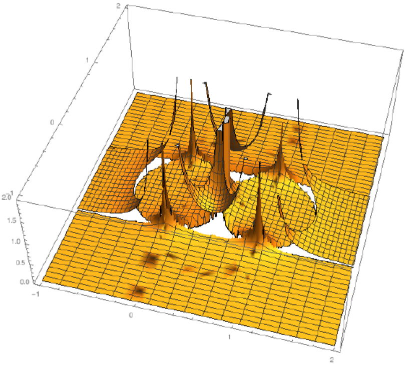
Define a dynamical system on Lorentz quadruples:

$$T(a,b,c,d) = \begin{cases} (2a-b-c-d, a-c-d, a-b-d, a-b-c) & 2a-b-c-d<0 \\ (2a-b+c+d, a+c+d, -a+b-d, -a+b-c) & 2a-b+c+d<0 \\ (2a+b+c-d, -a-c+d, -a-b+d, a+b+c) & 2a+b+c-d<0 \\ (2a+b-c+d, -a+c-d, a+b+d, -a-b-c) & 2a+b-c+d<0 \\ (2a-b-c+d, a-c+d, a-b+d, -a+b+c) & 2a-b-c+d<0 \\ (2a+b-c-d, -a+c+d, a+b-d, a+b-c) & 2a+b-c-d<0 \\ (2a-b+c-d, a+c-d, -a+b+d, a-b+c) & 2a-b+c-d<0 \\ (2a+b+c+d, -a-c-d, -a-b-d, -a-b-c) & 2a+b+c+d<0 \end{cases}$$

Theorem (Chaubey-Fuchs-Hines-S.)

This dynamical system moves to the root $(1,1,0,0)$ *along the swap-down tree.*

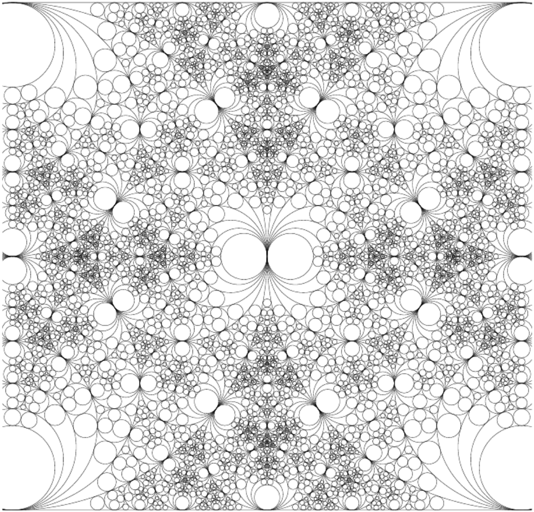
Invariant Measure



Theorem (Chaubey-Fuchs-Hines-S.)

There is an explicit invariant measure for T.

Words of length ≤ 5 in swap down tree

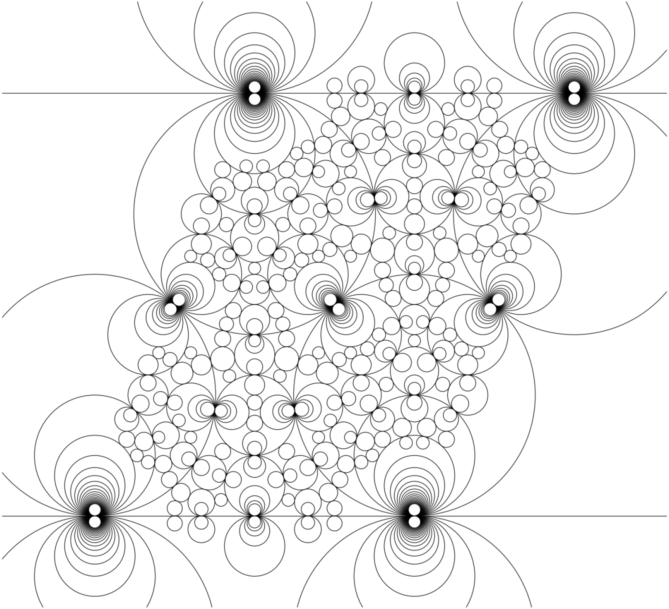


Circles generated by normal form words of length ≤ 5.

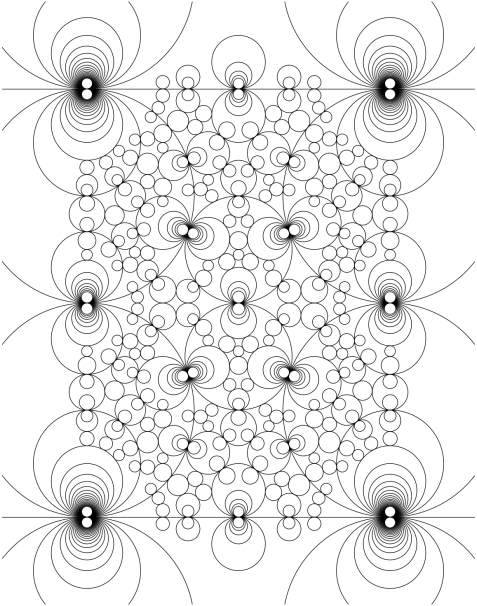
Comparison

$\mathbb{Z}$	$\mathbb{Z}[i]$
$\mathrm{PSL}_2(\mathbb{Z})$	$\mathrm{PSL}_2(\mathbb{Z}[i])$
$\{0, \infty\}$	$\widehat{\mathbb{R}}$
$\Pi(2)$	super-Apollonian group
superbasis (3 pts)	Descartes quadruple with dual (6 pts)
swap out one	swap out three
Stern-Brocot tree	swap-down or drop-down tree
Gauss map	apollonian dynamics

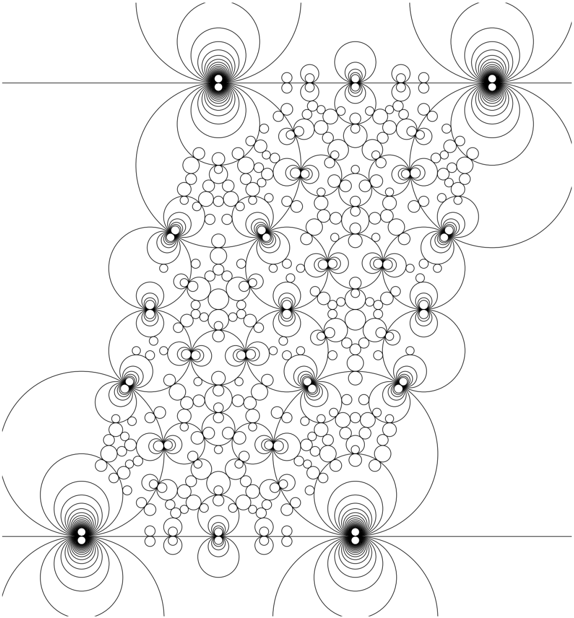
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-7})$



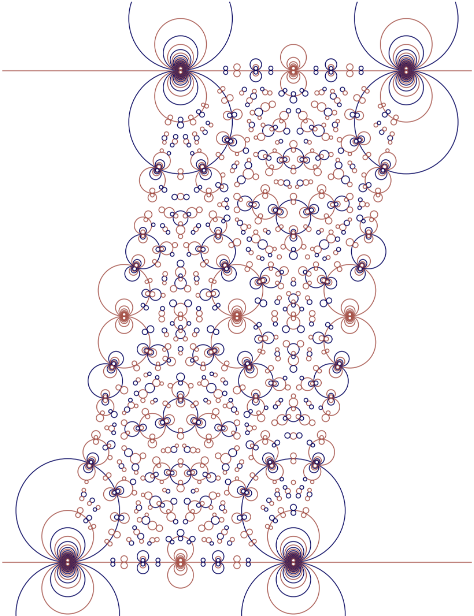
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-2})$



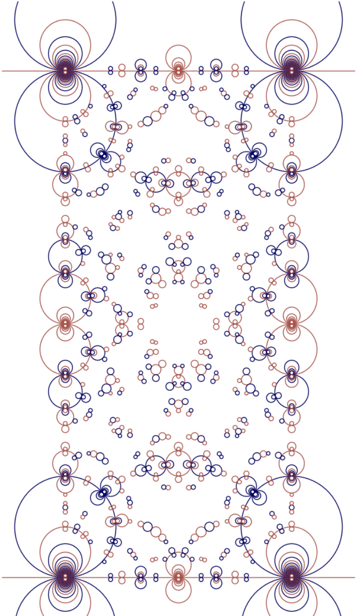
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-11})$



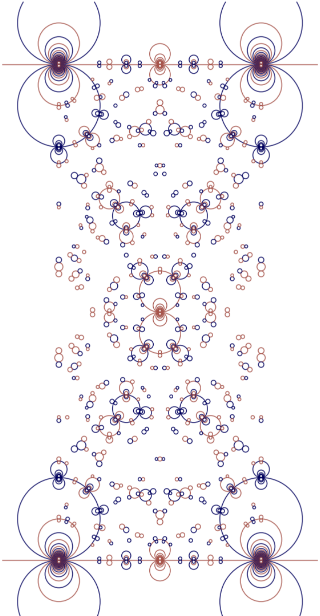
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-19})$



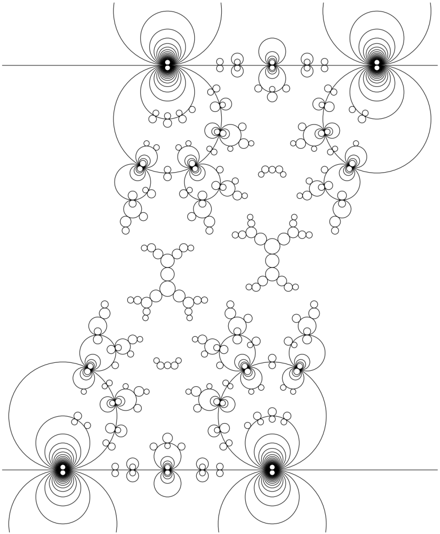
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-5})$



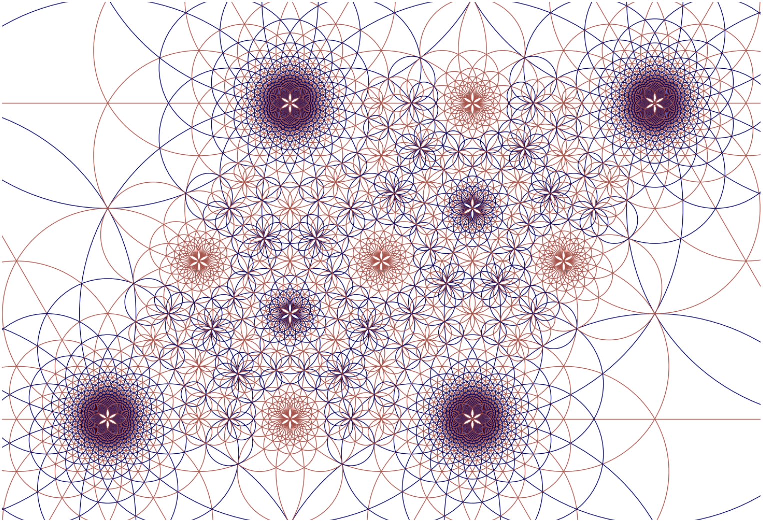
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-6})$



Schmidt Arrangement of $\mathbb{Q}(\sqrt{-15})$



Schmidt Arrangement of $\mathbb{Q}(\sqrt{-3})$



Now the theme on AMS YouTube, Twitter, etc.

Basic properties of $\mathcal{S}_K$

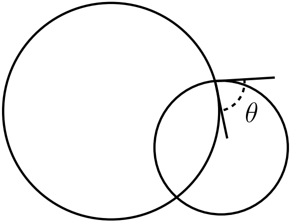
$$\Delta = \mathrm{Disc}(K)$$

Proposition (S.)

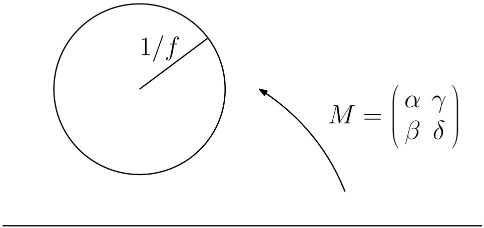
The curvatures in $\mathcal{S}_K$ lie in $\sqrt{-\Delta}\mathbb{Z}$.

Proposition (S.)

K-Bianchi circles intersect at points in K, at angles θ such that $e^{i\theta}$ is a unit in K.



Circles are ideal classes



Theorem (S.)

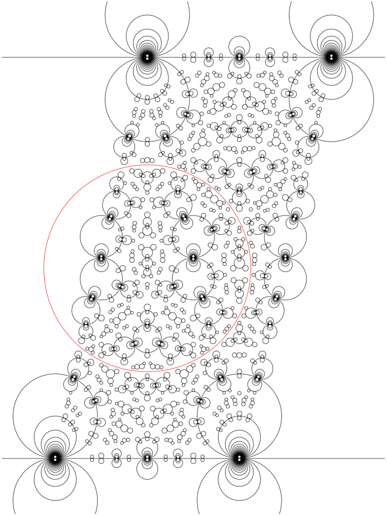
$$\left\{ \begin{array}{c} \textit{oriented} \\ \textit{circles} \end{array} \right\} \Big/ \left\{ \begin{array}{c} \textit{translations by } \mathcal{O}_K \textit{ and} \\ \textit{rotations by `unit angles'} \end{array} \right\} \qquad M(\widehat{\mathbb{R}}) \qquad f = \textit{curvature}$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$\left\{ \begin{array}{c} \textit{invertible} \\ \textit{ideal classes} \\ \mathfrak{a} \subset \mathcal{O}_f \end{array} \middle| \; f \in \mathbb{Z}^{>0}, \mathfrak{a}\mathcal{O}_K \sim \mathcal{O}_K \right\} \qquad \beta\mathbb{Z} + \delta\mathbb{Z} \qquad f = \textit{covolume}$$

Corollary: Number of circles of curvature f (up to equivalence) is h_f/h_K. (GLMWY for $\mathbb{Q}(i)$)

Euclideanity and $\mathcal{S}_K$



Theorem (S.)

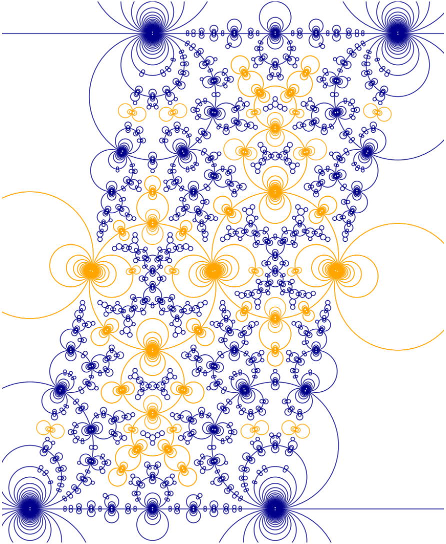
$\mathcal{S}_K$ is connected if and only if $\mathcal{O}_K$ is Euclidean.

The *ghost circle* is the circle orthogonal to the unit circle having center

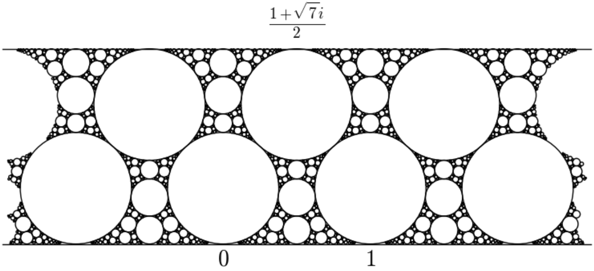
$$\begin{cases} \frac{1}{2} + \frac{\sqrt{\Delta}}{4} & \Delta \equiv 0 \pmod 4 \\ \frac{1}{2} + \frac{-\Delta-1}{4\sqrt{\Delta}} & \Delta \equiv 1 \pmod 4 \end{cases}.$$

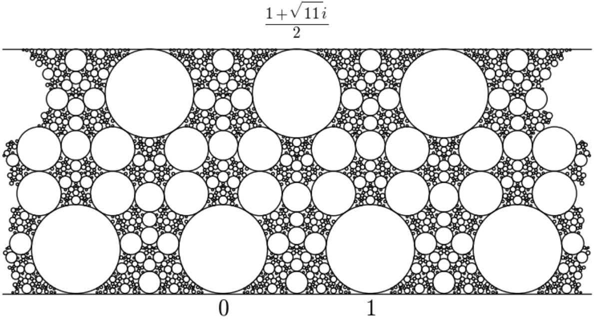
It exists only when $\mathcal{O}_K$ is non-Euclidean.

Schmidt Arrangement of $\mathbb{Q}(\sqrt{-15})$ with Ghost Circles

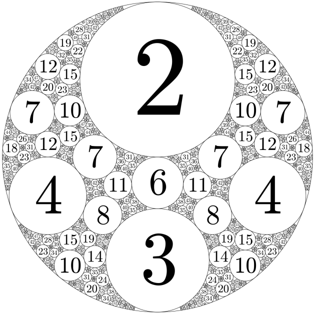


K-Apollonian Packings





K-Apollonian Packings



Theorem (S.)

The Schmidt arrangement is the disjoint union of all K-Apollonian circle packings (where circles are oriented).