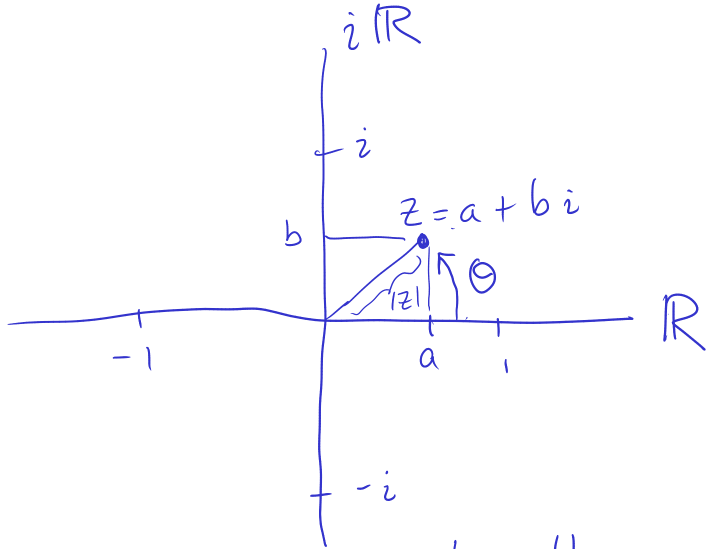
Complex Numbers

$\mathbb{C} = \mathbb{R} + i\mathbb{R} = \{ a+bi : a,b \in \mathbb{R} \}$ $\boxed{\text{a field w/ } +, \times}$



add: $(a+bi) + (c+di) = (a+c) + (b+d)i$

(addition of vectors)

multiplication: as if for polynomials but $i^2 = -1$

eg. $(2+3i)(5+7i)$

$= 10 + 15i + 14i + \underbrace{21i^2}_{-21}$

$= -11 + 29i$

length $z = a+bi$

$|z|^2 = a^2 + b^2$

$|z| = \sqrt{a^2+b^2}$

inverses $(a+bi)^{-1} = \dfrac{1}{a+bi} \cdot \dfrac{a-bi}{a-bi}$

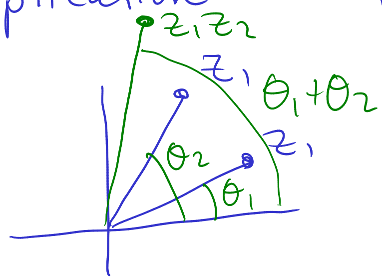
$= \dfrac{a-bi}{a^2+b^2} = \dfrac{a}{a^2+b^2} + \dfrac{-b}{a^2+b^2} i$

"polar" expression

$z = r \cdot e^{i\theta}$ $\quad r = \text{length} = |z|$

$\theta = \text{angle}$

Multiplication: $\underbrace{(r_1 e^{i\theta_1})}_{z_1} \underbrace{(r_2 e^{i\theta_2})}_{z_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$



aside:

$e^{a+bi} = e^a e^{ib}$

lengths multiply

angles add

$1 = 1 \cdot e^{i \cdot 0}$

$i = 1 \cdot e^{i\pi/2}$

$-1 = 1 \cdot e^{i\pi} \Rightarrow e^{i\pi} + 1 = 0$

Qubit

A classical bit is something which has two possible states: 0 and 1.

A qubit (quantum bit) is something whose possible states are of the form

$$\underbrace{a|0\rangle}_{\text{classical 0 state}} + \underbrace{b|1\rangle}_{\text{classical 1 state}} \qquad a, b \in \mathbb{C}, \quad |a|^2+|b|^2=1$$

"superposition" = linear combination of $|0\rangle, |1\rangle$
"pure states"

think of a length 1 complex vector in $\mathbb{C}^2$ $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$

Fact: $a|0\rangle + b|1\rangle$ and $e^{i\theta}(a|0\rangle + b|1\rangle)$ are physically indistinguishable.

$\uparrow$ $\mathbb{C}$ # of length 1

"global phase"

Essentially $\mathbb{P}^1_{\mathbb{C}}$.

Measurement

☆ with respect to a basis ($|0\rangle, |1\rangle$ for us)

variable name (like $y = a\binom{1}{0} + \binom{0}{1}$)

Measuring $|\psi\rangle = a|0\rangle + b|1\rangle$ w.r.t. basis $|0\rangle, |1\rangle$

it will return either $|0\rangle$ or $|1\rangle$

$$\begin{cases} |0\rangle & \text{w/ probability } |a|^2 \\ |1\rangle & \text{w/ prob. } |b|^2 \end{cases}$$

recall: $|a|^2 + |b|^2 = 1$

And then $|\psi\rangle = |0\rangle$ if it returned $|0\rangle$

$|\psi\rangle = |1\rangle$ if it returned $|1\rangle$

measuring "collapses" the superposition

Eg. $|\psi\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ msr in basis $|0\rangle, |1\rangle$

result	probability
$\lvert 0\rangle$	$1/2 = \lvert\frac{1}{\sqrt{2}}\rvert^2$
$\lvert 1\rangle$	$1/2$

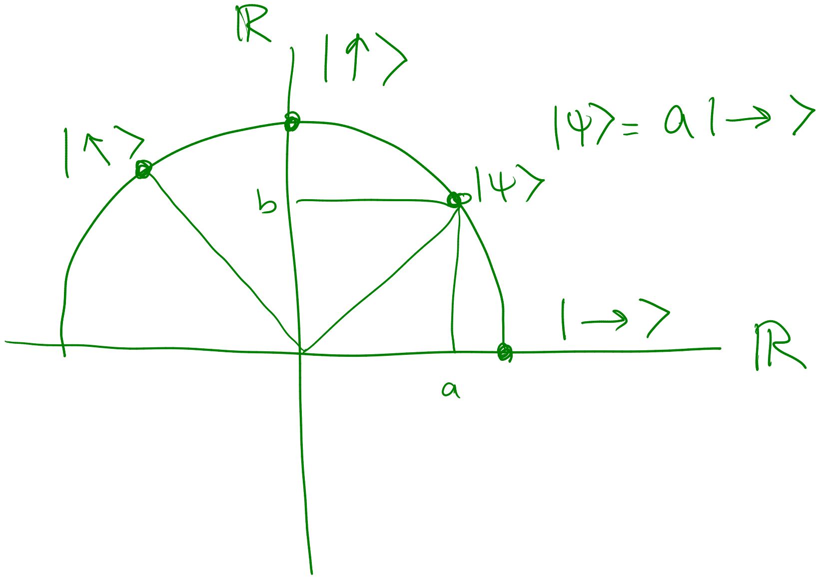
check: $\frac{1}{2} + \frac{1}{2} = 1$.

Multiple Bases

example w/ polarization of light

L basis $|\uparrow\rangle, |\rightarrow\rangle$

V basis $|\nearrow\rangle, |\nwarrow\rangle$



$$|\psi\rangle = a|\rightarrow\rangle + b|\uparrow\rangle$$

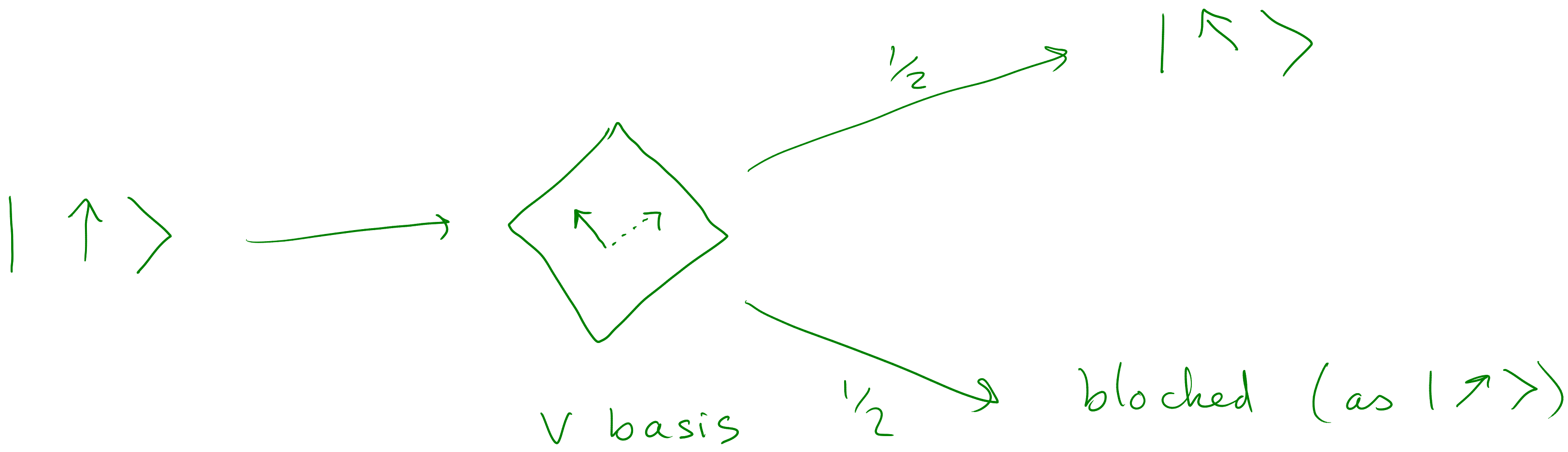
change of basis

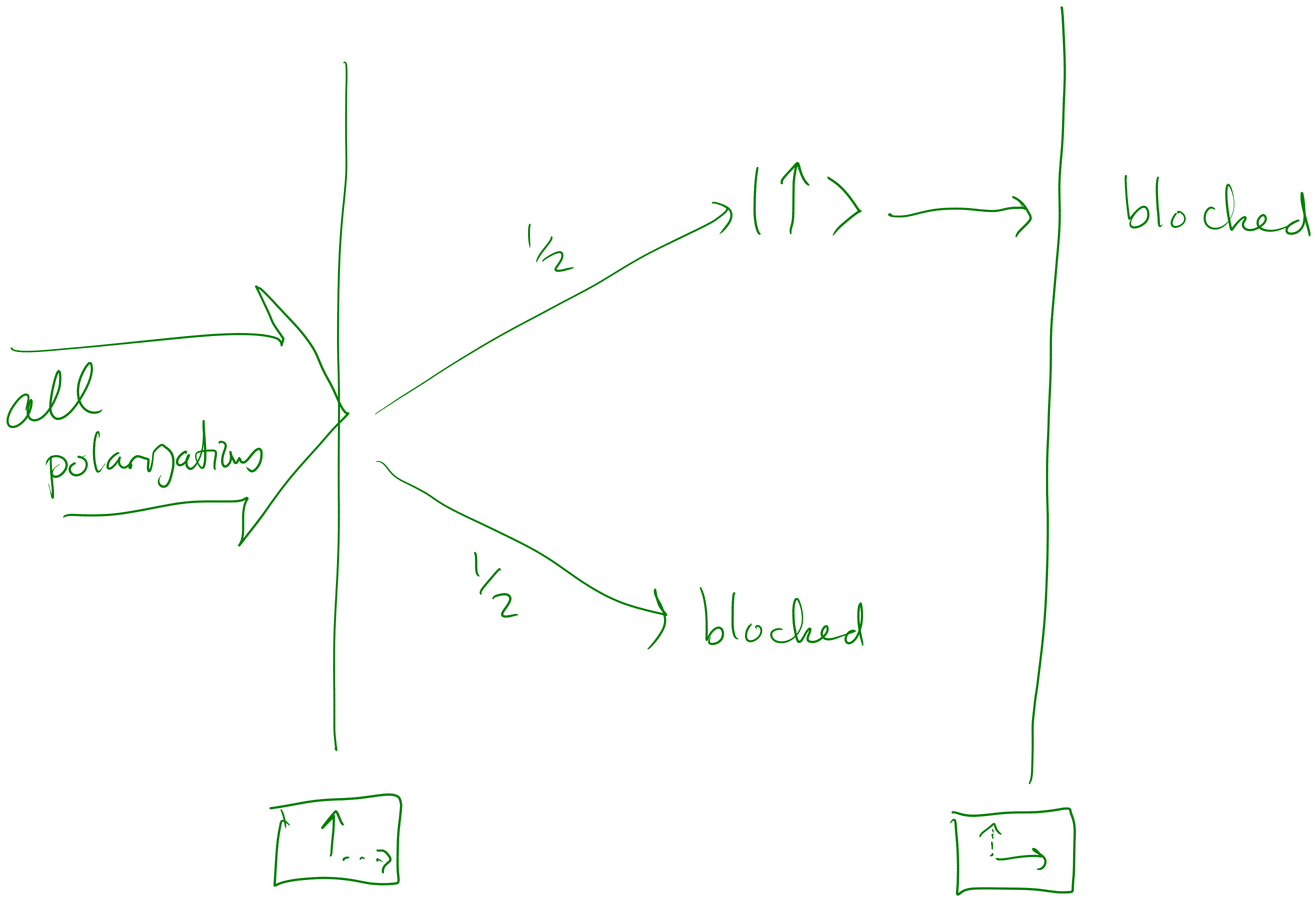
$$|\nwarrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\rightarrow\rangle$$

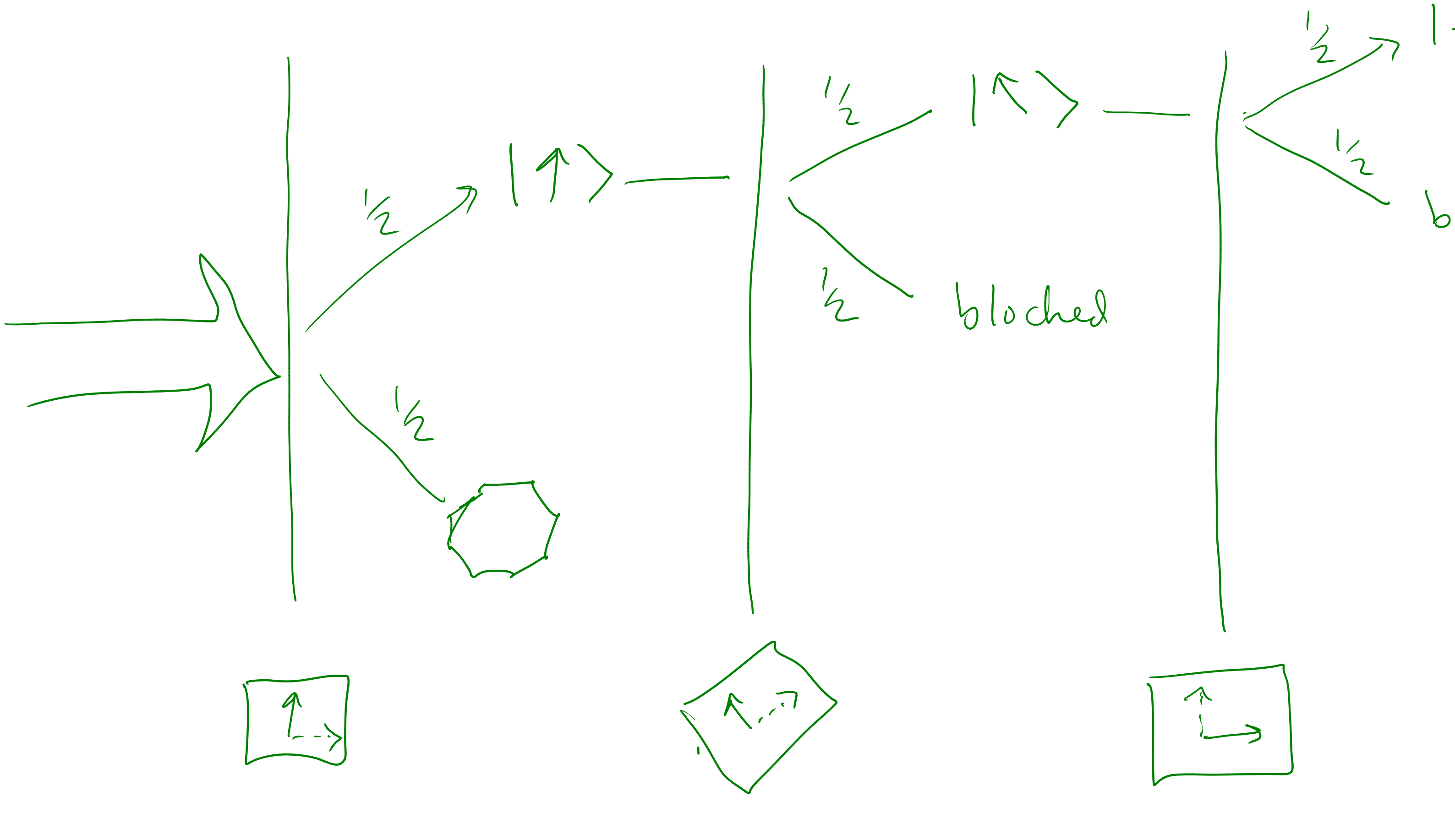
$$|\nearrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\rightarrow\rangle$$

$$|\uparrow\rangle = \frac{1}{\sqrt{2}}|\nearrow\rangle + \frac{1}{\sqrt{2}}|\nwarrow\rangle$$

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}|\nearrow\rangle - \frac{1}{\sqrt{2}}|\nwarrow\rangle$$


$|\uparrow\rangle$
$1/2$
$|\nwarrow\rangle$
V basis
$1/2$
blocked (as $|\nearrow\rangle$)


all polarizations
1/2
$|\uparrow\rangle$
blocked
1/2
blocked



$\frac{1}{8}$th of light gets through