Test Wednesday!

Qubits.

classical states 0 1

$$|\psi\rangle = a|0\rangle + b|1\rangle \quad , \quad a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1$$

measurement (wrt 0,1)	outcome	probab.
	$\|0\rangle$	$\|a\|^2$
	$\|1\rangle$	$\|b\|^2$

2 Qubits

00 01 10 11 classical states

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

notation $|1\rangle|1\rangle$

$$\alpha_{ij} \in \mathbb{C} \quad , \quad |\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$$

measure full 2qubit system (wrt 00, 01, 10, 11)	result	prob
	$\|00\rangle$	$\|\alpha_{00}\|^2$
	$\|01\rangle$	$\|\alpha_{01}\|^2$
	$\|10\rangle$	$\|\alpha_{10}\|^2$
	$\|11\rangle$	$\|\alpha_{11}\|^2$

Example to introduce 1 qubit measurement

(*) eg. $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle = |0\rangle\left(\frac{1}{\sqrt{2}}|0\rangle\right) + |1\rangle\left(\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle\right)$

measure just 1st bit	bit read	outcome state	probabilities
	0	$\|0\rangle\|0\rangle$	$\left\|\frac{1}{\sqrt{2}}\right\|^2 = \frac{1}{2}$
	1	$\sqrt{2}\,\|1\rangle\left(\frac{1}{2}\|0\rangle + \frac{1}{2}\|1\rangle\right) = \frac{1}{\sqrt{2}}\|10\rangle + \frac{1}{\sqrt{2}}\|11\rangle$	$\left\|\frac{1}{2}\right\|^2 + \left\|\frac{1}{2}\right\|^2 = \frac{1}{2}$

renormalize

In general.

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

Msr 1st qubit (in basis 0,1):

Result Bit	Prob	State after Measurement
0	$\|\alpha_{00}\|^2 + \|\alpha_{01}\|^2$	$(\alpha_{00}\|00\rangle + \alpha_{01}\|01\rangle)\,(\|\alpha_{00}\|^2 + \|\alpha_{01}\|^2)^{-1/2}$
1	$\|\alpha_{10}\|^2 + \|\alpha_{11}\|^2$	$(\alpha_{10}\|10\rangle + \alpha_{11}\|11\rangle)\,(\|\alpha_{10}\|^2 + \|\alpha_{11}\|^2)^{-1/2}$

Then msr 2nd qubit:

1st qubit	2nd qubit	Prob (of total msrment)	State outcome
0	0	$\|\alpha_{00}\|^2$	$\|00\rangle$
	1	$\|\alpha_{01}\|^2$	$\|01\rangle$
1	0	$\|\alpha_{10}\|^2$	$\|10\rangle$
	1	$\|\alpha_{11}\|^2$	$\|11\rangle$

Bottom Line: Order of measurement doesn't matter.

Bell State $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ (entangled)

Unentangled system of 2 qubits:

$$|\psi_1\rangle = a_1|0\rangle + b_1|1\rangle$$
$$|\psi_2\rangle = a_2|0\rangle + b_2|1\rangle$$

as a joint state: $(a_1|0\rangle + b_1|1\rangle)(a_2|0\rangle + b_2|1\rangle)$

$$= a_1a_2|00\rangle + a_1b_2|01\rangle + a_2b_1|10\rangle + b_1b_2|11\rangle$$

Anything not of this form is <u>entangled</u>.

Claim: Bell State is entangled.

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$$

If it were unentangled, the system

$a_1a_2 = \frac{1}{\sqrt{2}}$ $\quad a_1b_2 = 0$ $\quad a_2b_1 = 0$ $\quad b_1b_2 = \frac{1}{\sqrt{2}}$

would have a solution.

trick
$(a_1b_2)(a_2b_1)$
$= (a_1a_2)(b_1b_2)$

By #2: $a_1 = 0$ or $b_2 = 0$
$\Rightarrow a_1a_2 = 0$ $\rightarrow\leftarrow$ $\quad \Rightarrow b_1b_2 = 0$ $\rightarrow\leftarrow$ } NO SOLUTION.

Another eg. $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$ entangled.

$|00\rangle = (|0\rangle)(|0\rangle)$ unentangled

(*) entangled

if $\alpha_{01}\alpha_{10} \neq \alpha_{00}\alpha_{11}$ then it is entangled

More qubits? n qubits

$\rightarrow$ superposition of 2^n classical states

measurement order doesn't matter
& works same way

entangled / not same idea

Eg. Quantum Key Distributions using Entanglement (Ekert '91)

Alice $\xleftarrow{\text{1st qubit}}$ source of Bell states $\xrightarrow{\text{2nd qubit}}$ Bob

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Bases (Alice)

$\pi/4$	A_2
$\pi/8$	A_1
0	A_0

Bases (Bob)

$\pi/8$	B_2
0	B_1
$-\pi/8$	B_0

Both msr w/ random bases:

Prob of $\frac{2}{9}$ that bases agree $\Rightarrow$ results agree $\Rightarrow$ key

Remaining data: use "Bell test" to check entanglement