Classical Computer

bits
gates

Example.

NOT

in	out
0	1
1	0

AND

in	out
00	0
01	0
10	0
11	1

NAND

in	out
00	1
01	1
10	1
11	0

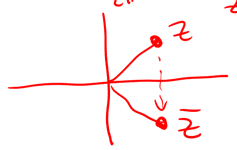
can compute any f^n

$f: \{0,1\}^n \to \{0,1\}^k$

Quantum Computer

qubits
quantum gates
= unitary operators

Conjugation: $z = a+bi$
$\bar{z} = a-bi$



refl. across R axis

Def^n. A $n\times n$ $\mathbb{C}$ matrix $\mathcal{U}$ is unitary if

$\mathcal{U}\mathcal{U}^\dagger = I = \mathcal{U}^\dagger \mathcal{U}$ where $\mathcal{U}^\dagger$ is the conjugate transpose of $\mathcal{U}$

eg. $\mathcal{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\mathcal{U}^\dagger = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$

swap rows & columns & conjugate

In geometric terms, $\mathcal{U}$ is unitary if $|\mathcal{U}\vec{v}| = |\vec{v}|$ for all $\vec{v} \in \mathbb{C}$

$|(v_1, \dots, v_n)| = \sqrt{|v_1|^2 + |v_2|^2 + \dots + |v_n|^2}$

Examples (of unitary transformations/matrices)

1-qubit quantum gates

In basis $|0\rangle, |1\rangle$
States of 1-qubit: $a|0\rangle + b|1\rangle$, $a,b \in \mathbb{C}$
$|a|^2 + |b|^2 = 1$

think of as a vector in $\mathbb{C}^2$ i.e. $\begin{pmatrix} a \\ b \end{pmatrix}$ of length 1

name	matrix	conj. transp.	$\mathcal{U}\mathcal{U}^\dagger$ (check unitary)	behaviour
NOT Pauli X	$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$ $\|0\rangle \longmapsto \|1\rangle$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\|1\rangle \longmapsto \|0\rangle$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $a\|0\rangle + b\|1\rangle \longmapsto b\|0\rangle + a\|1\rangle$.
Pauli Z	$Z = \begin{array}{cc} \|0\rangle & \|1\rangle \end{array}$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{array}{c} \|0\rangle \\ \|1\rangle \end{array}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\|0\rangle \longmapsto \|0\rangle$ $\|1\rangle \longmapsto -\|1\rangle$
Pauli Y	$Y = \begin{array}{cc} \|0\rangle & \|1\rangle \end{array}$ $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{array}{c} \|0\rangle \\ \|1\rangle \end{array}$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\|0\rangle \longmapsto i\|1\rangle$ $\|1\rangle \longmapsto -i\|0\rangle$

check: $\frac{(1+i)}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle \longmapsto \frac{-i}{\sqrt{3}}|0\rangle + \frac{i-1}{\sqrt{3}}|1\rangle$

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1+i \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ i-1 \end{pmatrix}\frac{1}{\sqrt{3}}$$

Hadamard $\quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad HH^{\dagger} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓ unitary

behaviour: $|0\rangle \longmapsto \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$|1\rangle \longmapsto \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Two-qubit gates. basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

4×4 matrix

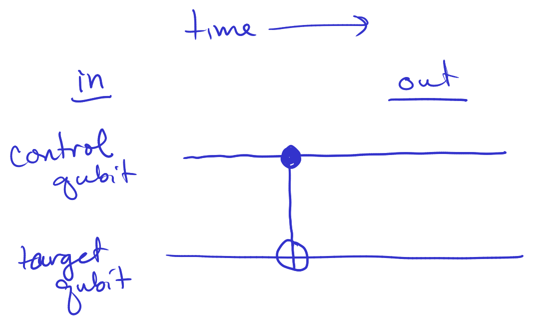
CNOT gate

$$\begin{array}{c|c} & \begin{matrix} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{matrix} \\ \hline & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix} \end{array}$$

$$\left.\begin{aligned} |00\rangle &\longmapsto |00\rangle \\ |01\rangle &\longmapsto |01\rangle \\ |10\rangle &\longmapsto |11\rangle \\ |11\rangle &\longmapsto |10\rangle \end{aligned}\right\}$$

1st bit is "control bit" — controls whether you "Not"

2nd bit is "target bit" — the object of your "Not"ting

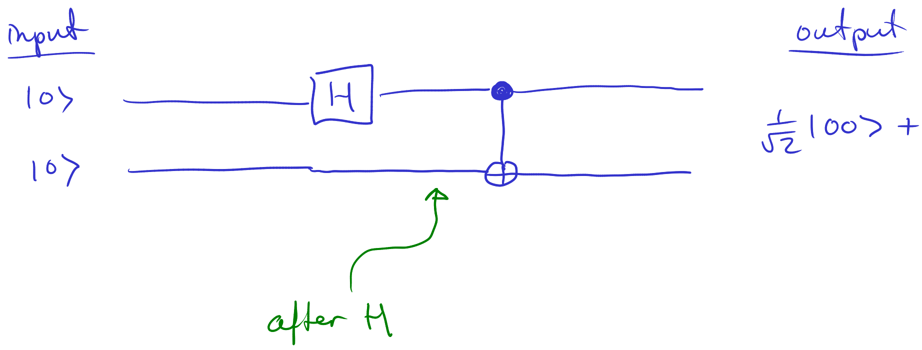


Check: apply CNOT to $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$

answer: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Circuit to create a $\boxed{\text{Bell state}}$ from $\boxed{|00\rangle}$

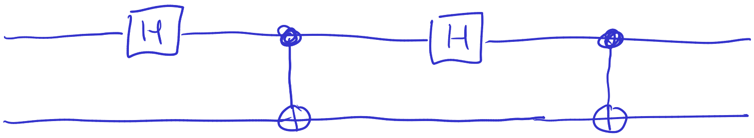


$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

$$|0\rangle|0\rangle \xrightarrow{\boxed{H}} \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)|0\rangle$$

$$= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Check what happens if you do this twice?



$$|00\rangle \longmapsto \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \overset{H}{\longmapsto} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|11\rangle\right)$$

$$= \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$$

H on 1st of 2 qubits

$$\begin{array}{cc} & \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\ \frac{1}{\sqrt{2}} & \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array}$$

$$\boxed{|00\rangle \longmapsto \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle}$$

$$|01\rangle \longmapsto \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$|10\rangle \longmapsto \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

$$\boxed{|11\rangle \longmapsto \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|11\rangle}$$

$$\xrightarrow{\text{CNOT}} \frac{1}{2}\left(|00\rangle + |01\rangle - |10\rangle + |11\rangle\right)$$