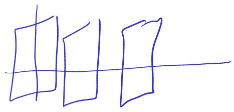
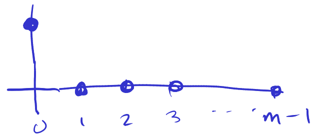
The space of functions $\{ f : \mathbb{Z}/_m\mathbb{Z} \to \mathbb{C} \}$



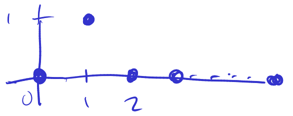
## Standard Basis

$f_0$



$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$f_1$



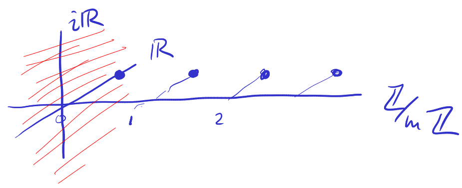
$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\vdots$

$$\omega_m = e^{\frac{2\pi i}{m}}$$

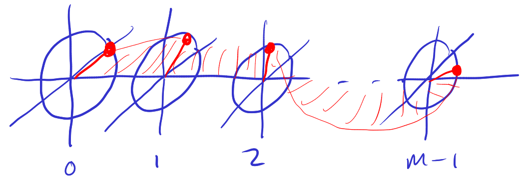
## Fourier Basis

freq = 0



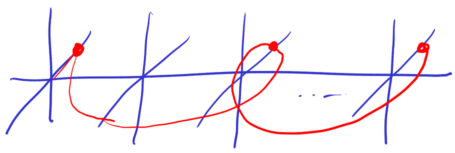

$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

freq = 1

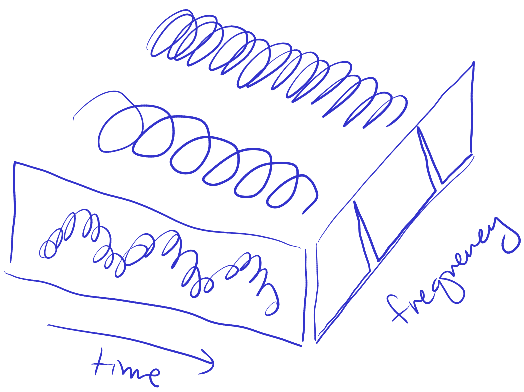



$\begin{pmatrix} 1 \\ \omega_m \\ \omega_m^2 \\ \vdots \\ \omega_m^{m-1} \end{pmatrix}$

freq = 2



$\begin{pmatrix} 1 \\ \omega_m^2 \\ \omega_m^4 \\ \vdots \\ \omega_m^{2m-2} \end{pmatrix}$




## The Quantum Fourier Transform

$\omega = \omega_m$

$$F_m = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{m-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2m-2} \\ \vdots & \omega^3 & \omega^6 & \cdots & \omega^{3m-3} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{m-1} & \omega^{2m-2} & \cdots & \omega^{(m-1)(m-1)} \end{pmatrix}$$

columns = fourier basis elements written in terms of standard basis coords.

$m \times m$ matrix <u>unitary</u>

eg. <u>$m = 2$</u>

1-qubit gate

$$F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Hadamard.

$\omega_2 = -1$

eg. $m = 4$

2-qubit gate

$$F_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

basis $|00\rangle$ $|01\rangle$ $|10\rangle$ $|11\rangle$

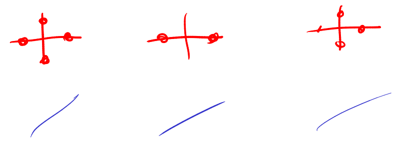
$\omega_4 = i$

$$|00\rangle \longmapsto \tfrac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$|01\rangle \longmapsto \tfrac{1}{2}\left(|00\rangle + i|01\rangle - |10\rangle - i|11\rangle\right)$$

$$|10\rangle \longmapsto \tfrac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)$$

$$|11\rangle \longmapsto \tfrac{1}{2}\left(|00\rangle - i|01\rangle - |10\rangle + i|11\rangle\right)$$



$$\tfrac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \longmapsto |00\rangle$$

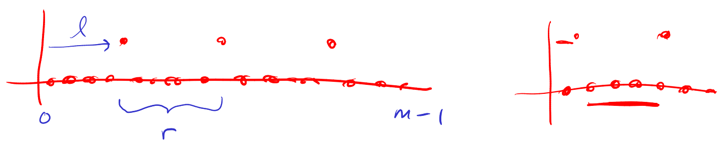
## QFT.

$F_m$ is unitary.

$$\sum_{x=0}^{m-1} f(x)|x\rangle \longmapsto \sum_{x=0}^{m-1} \hat{f}(x)|x\rangle$$

$$= \frac{1}{\sqrt{m}} \sum_{x=0}^{m-1} \left(\sum_{k=0}^{m-1} \omega_m^{xk} f(k)\right)|x\rangle$$

## QFT on a periodic state



$m = 2^n$ ($n$ qubits)

input: $$\frac{1}{\sqrt{m/r}} \sum_{k=0}^{m/r - 1} |\ell + kr\rangle$$

include (value 1 for coeff) kets w/ $x \equiv \ell \pmod{r}$

$r \mid m$

$$\xrightarrow{\text{QFT}} \frac{1}{\sqrt{m}\sqrt{m/r}} \sum_{y=0}^{m-1} \underbrace{\left(\sum_{k=0}^{m/r-1} \omega_m^{(\ell+kr)y}\right)}_{\circledast}|y\rangle$$

$$\circledast = \begin{cases} y \in \frac{m}{r}\mathbb{Z} \to \omega_m^{\ell y} \displaystyle\sum_{k=0}^{m/r-1} \omega^{kry} = \omega_m^{\ell y} \sum_{k=0}^{m/r-1} 1 = \omega_m^{\ell y} \cdot \frac{m}{r} \\ y \notin \frac{m}{r}\mathbb{Z} \to 0 \end{cases}$$


QFT
M/r
2M/r
3M/r