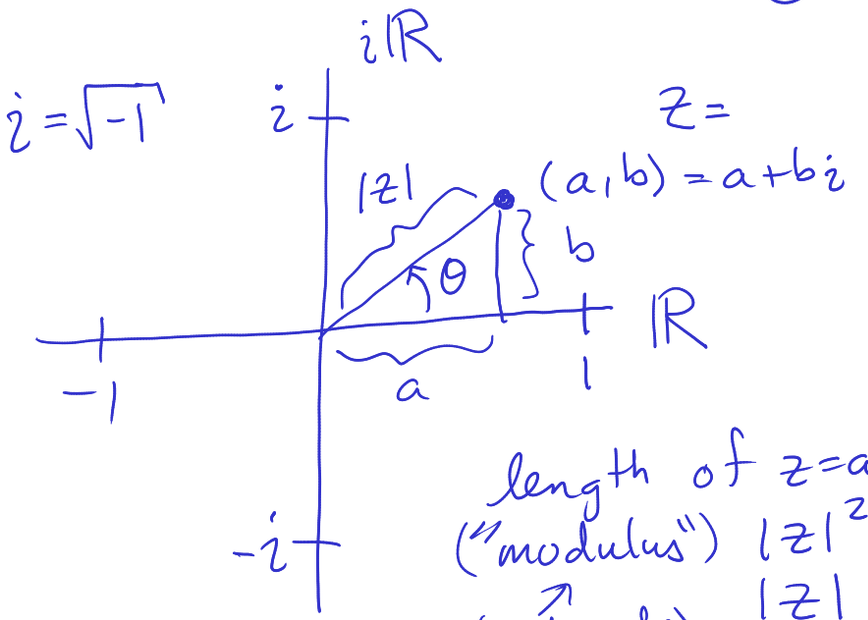
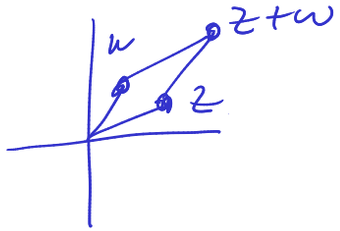
Complex Numbers

$\mathbb{C} = \mathbb{R} + i\mathbb{R} = \{ a+bi : a,b \in \mathbb{R} \}$ a field w/ $+, \times$



add: $(a+bi)+(c+di) = (a+c)+(b+d)i$

(addition of vectors)



multiplication: as for poly in i with $\underline{i^2 = -1}$.

length of $z=a+bi$ ("modulus") $|z|^2 = a^2+b^2$

(not modn) $|z| = \sqrt{a^2+b^2}$

eg. $(2+3i)(5+7i)$

$= 10 + 15i + 14i + 21i^2$

$= 10 + 29i - 21 = -11 + 29i$

inverses: $(a+bi)^{-1} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2}$

$= \left(\frac{a}{a^2+b^2}\right) + \left(\frac{-b}{a^2+b^2}\right) i$

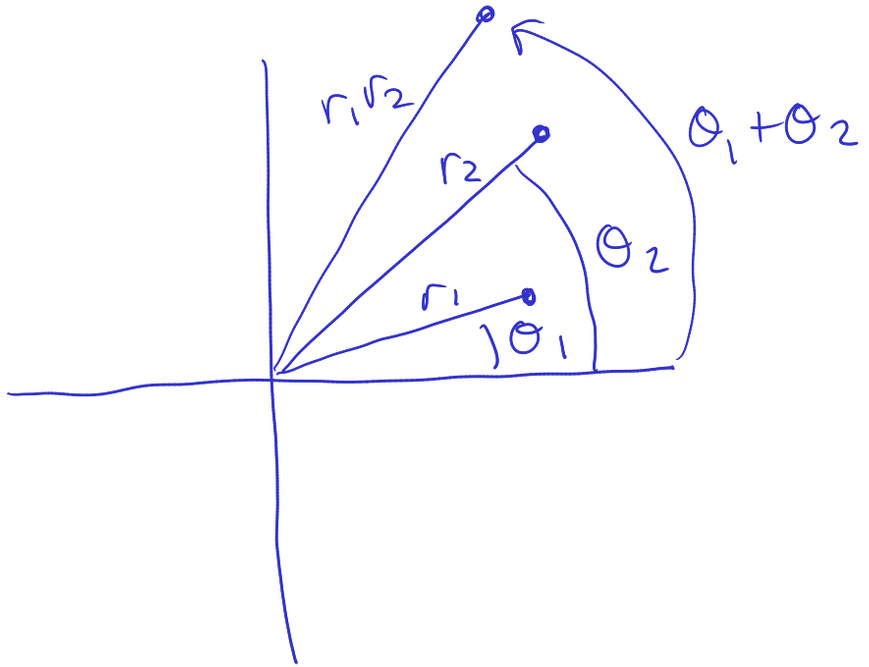
the <u>conjugate</u> of $a+bi$ is $a-bi$

"polar" expression

$z = r \cdot e^{i\theta}$ r = length $= |z|$, θ = angle

$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2})$

$= r_1 r_2 e^{i(\theta_1+\theta_2)}$

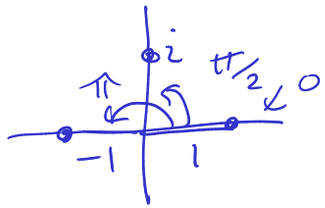


lengths multiply
angles add

$$1 = 1 \cdot e^{i \cdot 0}$$

$$i = 1 \cdot e^{i\frac{\pi}{2}}$$

$$-1 = 1 \cdot e^{i\pi}$$



$$e^{a+bi} = e^a e^{ib}$$

$$i \cdot i = 1 \cdot e^{i\frac{\pi}{2}} \cdot 1 \cdot e^{i\frac{\pi}{2}}$$

$$= 1 \cdot e^{i\pi} = -1$$

Qubit

A classical bit is something which has two possible states: 0 or 1.

A qubit (quantum bit) is something whose possible states are of the form

$$a|0\rangle + b|1\rangle, \quad a,b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1$$

($|0\rangle$: classical state 0; $|1\rangle$: classical state 1)

i.e. a length 1 complex vector in $\mathbb{C}^2$ $\quad \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$

"superposition" = linear combination of $|0\rangle, |1\rangle$ ("pure states")

Fact: $a|0\rangle + b|1\rangle$ and $e^{i\theta}(a|0\rangle + b|1\rangle)$ are physically indistinguishable

"global phase" — $e^{i\theta} \in \mathbb{C}$ number of length 1

Essentially $\mathbb{P}\mathbb{C}'$.

Measurement

variable name

Measuring $|\psi\rangle = a|0\rangle + b|1\rangle$ with respect to basis $|0\rangle, |1\rangle$ will return either $|0\rangle$ or $|1\rangle$, with probability:

$$\begin{cases} |0\rangle & \text{w/ probability } |a|^2 \\ |1\rangle & \text{w/ probability } |b|^2 \end{cases}$$

and then $|\psi\rangle = |0\rangle$ if it returned $|0\rangle$
$|\psi\rangle = |1\rangle$ " " $|1\rangle$.

measuring "collapses" the superposition.

Eg. $|\psi\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ Measure in basis $|0\rangle, |1\rangle$.

psi

result	probability
$\|0\rangle$	$\left\|\frac{i}{\sqrt{2}}\right\|^2 = \frac{1}{2}$
$\|1\rangle$	$\left\|\frac{1}{\sqrt{2}}\right\|^2 = \frac{1}{2}$

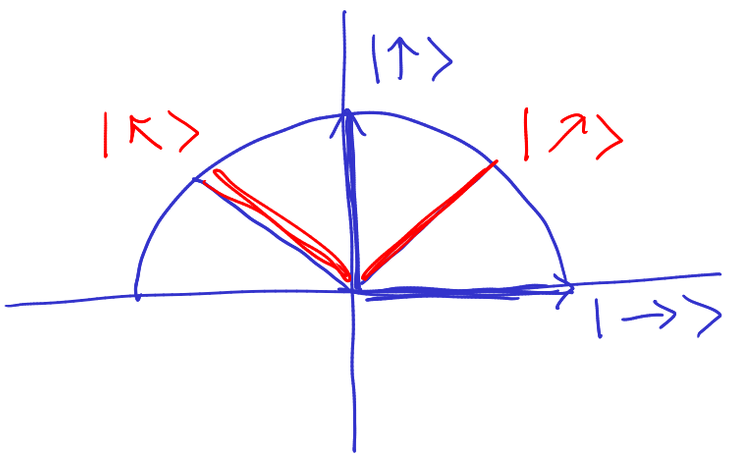
check: $\frac{1}{2} + \frac{1}{2} = 1$

Multiple Bases

example w/ polarization of light

L basis $|\uparrow\rangle, |\rightarrow\rangle$

V basis $|\nearrow\rangle, |\nwarrow\rangle$



change of basis

$$|\nwarrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\rightarrow\rangle$$

$$|\nearrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\rightarrow\rangle$$

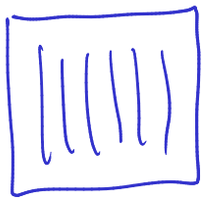
$$|\uparrow\rangle = \frac{1}{\sqrt{2}}|\nearrow\rangle + \frac{1}{\sqrt{2}}|\nwarrow\rangle$$

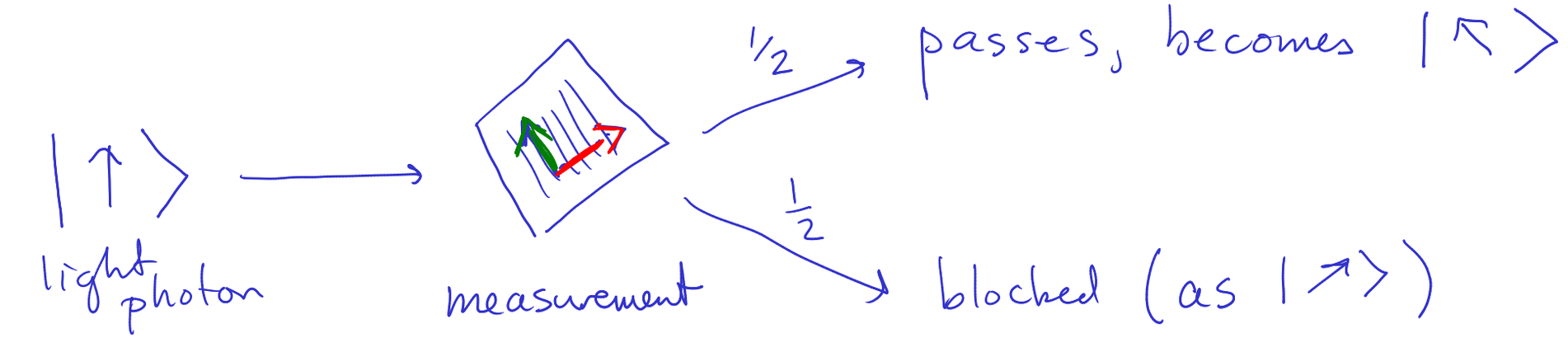
$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}|\nearrow\rangle - \frac{1}{\sqrt{2}}|\nwarrow\rangle$$

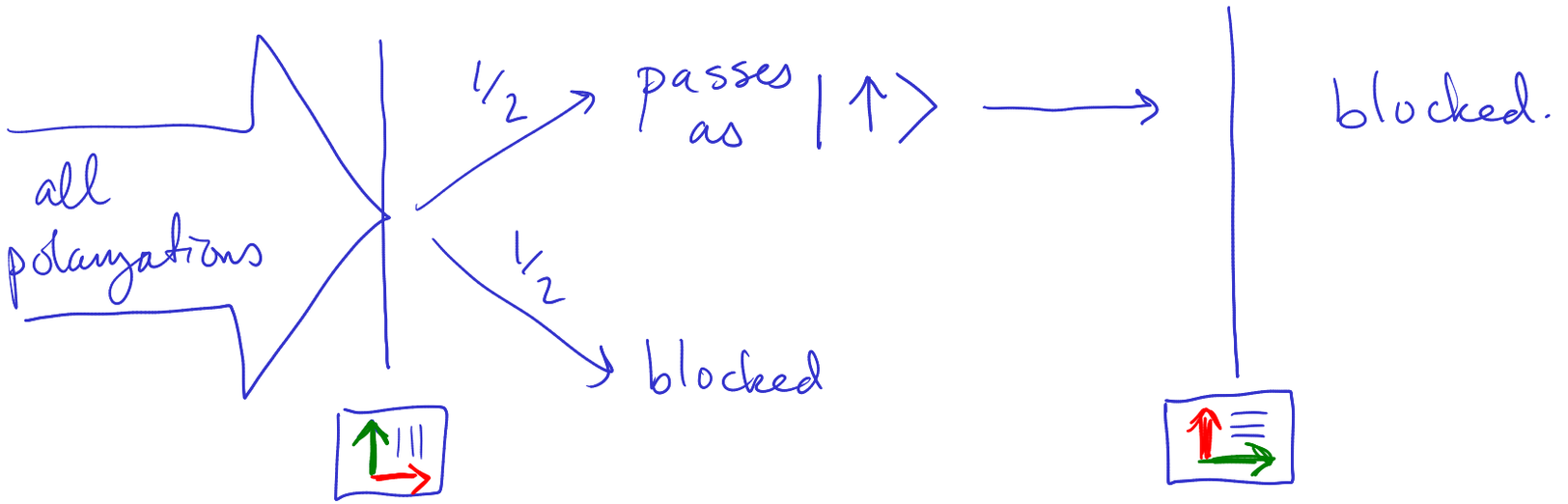
normalized

$$\left(\left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = 1\right)$$

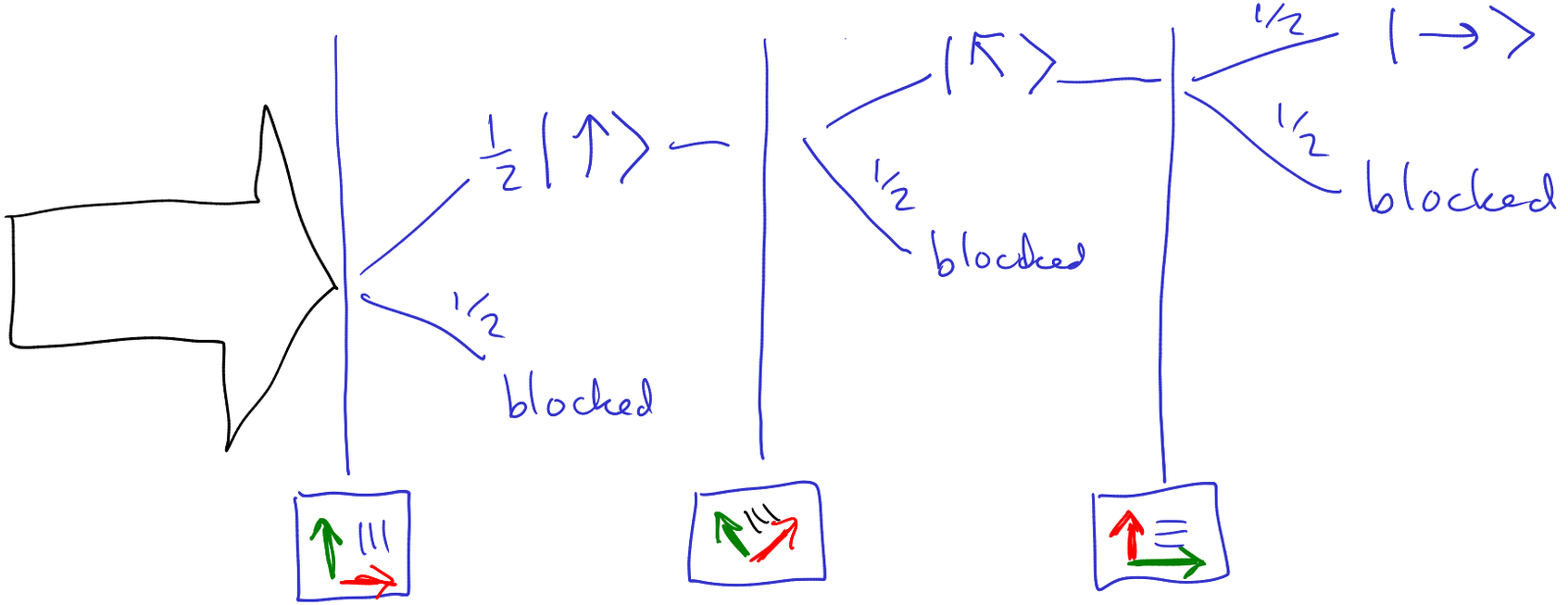
Polarized Filter







no light gets through



$\frac{1}{8}^{th}$ of light gets through